Box 14

Gauss's Geometrical Approach to Algebra

As Gauss devastatingly exposes in his 1799 doctoral dissertation, the approach to algebra as being ontologically arithmetic fails to explain itself: Algebra fails, internally, to prove what became known as the fundamental theorem of algebra.[1]

To clarify, consider Gauss's description of d'Alembert:

"It is proper to observe, that d'Alembert applied geometric considerations in the exposition of his proof and looked upon X as the abscissa, and x as the ordinate of a curve (according to the custom of all mathematicians of the first part of this century to whom the notion of functions was less familiar). But all his reasoning, if one considers only what is essential, rests not on geometric but on purely analytic principles, and an imaginary curve and imaginary ordinates are rather hard concepts and may offend a reader of our time. Therefore I have rather given here a purely analytic form of representation. This footnote I have added so that someone who compares d'Alembert's proof with this concise exposition may not mistrust that anything essential has been altered."

Compare this with Gauss's presentation of the ontologically geometric complex domain.

Gauss begins the portion of his dissertation concerning his own demonstration with two introductory lemmas, where he introduces two equations:

$$(1) \quad r^m \cos m\varphi + Ar^{(m-1)} \cos(m-1)\varphi + Br^{(m-2)} \cos(m-2)\varphi + \ldots + Krr \cos 2\varphi + Lr \cos\varphi + M = 0,$$

$$(2) \quad r^m \sin m\varphi + Ar^{(m-1)} \sin(m-1)\varphi + Br^{(m-2)} \sin(m-2)\varphi + \ldots + Krr \sin 2\varphi + Lr \sin\varphi + M = 0,$$

He then begins his proof proper:

"The outstanding theorem is frequently proved with the help of imaginary numbers, cf. Euler *Introd. In Anal. Inf. T.I. p 110;* I consider it worth the trouble to show how it can easily be elicited without their help. It is quite manifest that for the proof of our theorem nothing more is required than to show: *When any function X of the form $x^m + Ax^{(m-1)} + Bx^{(m-2)} +$ etc. $+ Lx + M$ is given, then r and φ can be determined in such a way that the equations (1) and (2) hold.*"

Not only does he claim that he will not use imaginary numbers, but he seems not even to use algebra! These equations (1) and (2) do not involve x in any way, but only r and φ.

To understand Gauss's use of these two equations (1) and (2), let's reapproach our earlier paradox, introduced in Box 13 (**Figure 1**):

We have lines, squares with one mean, and cubes with two means. What form could correspond to a greater number of means, or an indeterminate number of means? What Jakob Bernoulli reported as his *spira mirabilis* (miraculous spiral) provides us a lead (**Figure 2**).

Such a spiral combines two forms of action, known as arithmetic (simple, repeated addition) and geometric (simple, repeated multiplication). The amount of arithmetic angular change and geometric increase of distance are combined as one action: Thus, doubling the rotation squares the multiplied length, tripling cubes it, and quadrupling gives us a geometric understanding of x^4, x^5, x^6, and so on, as high as you like.

The unbridgeable gap between linear, square, and cubic action, and the mystery of higher forms of action, have been solved by introducing a single curve, which, by multiplying the amount of rotation, can create all of these relationships. Thus the equiangular spiral brings what seemed infinite, to the finite, and encompasses a before-then disparate class under one idea of action, which action Leibniz called logarithmic.

Now, there are many spirals that could be drawn, spirals which grow more or less quickly. Let us interest ourselves in the extremes: a straight line (pure extension, without rotation) and a circle (pure rotation, without extension) (**Figure 3**):

Inspect the circle (**Figure 4**): What form of number does it require? Call one location 1, and, naturally, its opposite -1:

Note that our earlier spiral relationship still holds: The 180° rotation to get to -1, when doubled to 360°, puts us at 1, which is $(-1)^2$. But what of the other locations on the circle? To what numbers do they correspond? They cannot all be 1, for they are different places (**Figure 5**).

Maintaining our principle, $(?)^2$ would be -1 by the logarithmic property doubling rotation on our spiral. This makes $(?) = \sqrt{-1}$, and its opposite, $-\sqrt{-1}$ (**Figure 6**).

The "imaginary" numbers, although

FIGURE 1

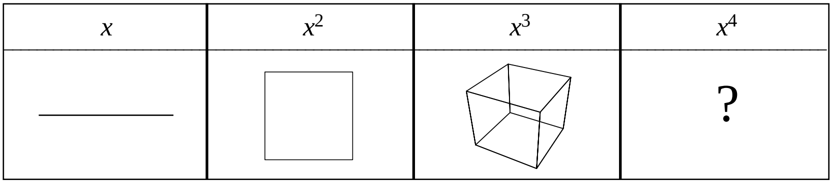
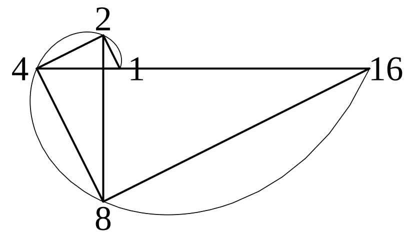


FIGURE 2



Bernoulli's logarithmic, self-similar spiral.[2] *The 90° rotation of going from 1 to 2, repeated four times to 360°, gives a length of 16, which is* 2^4.

FIGURE 3

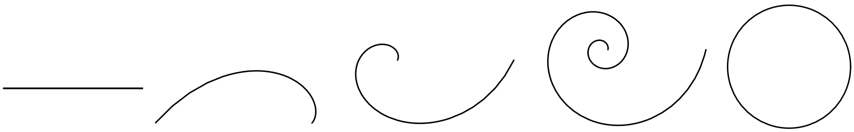


FIGURE 4

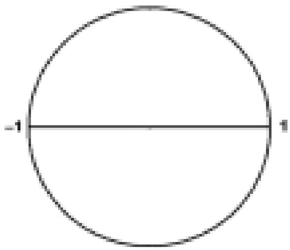


FIGURE 5

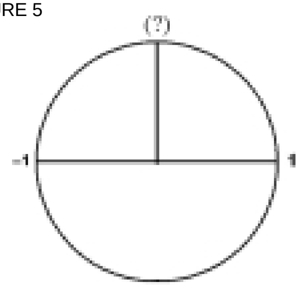
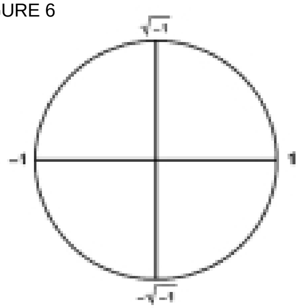


FIGURE 6



not existing on the number line, *do* exist, lying outside the blinders of formalists. Extending these actions, we create the complex domain.

"Suppose, however, the objects are of such a nature that they cannot be ordered in a single series, even if unboundedly in both directions, but can be ordered only in a series of series or, in other words, form a manifold of two dimensions; if the relation of one series to another or the transition from one series to another occurs in a similar manner, as we earlier described for the transition from a member of one series to another member of the same series, then in order to measure the transition from one member of the system to another, we shall require in addition to the already introduced units +1 and −1 two additional, opposite units $+i$ and $-i$. Clearly we must also postulate that the unit i [$\sqrt{-1}$ –ed.] always signifies the transition from a given member to a *determined* member of the immediately adjacent series. In this manner the system will be doubly ordered into a series of series."[3]

Now, how can we represent change in this complex domain? With "normal" numbers, squaring can be represented thus (**Figure 7**):

Each of these right angles combined

Box 14 continues on next page

FIGURE 7

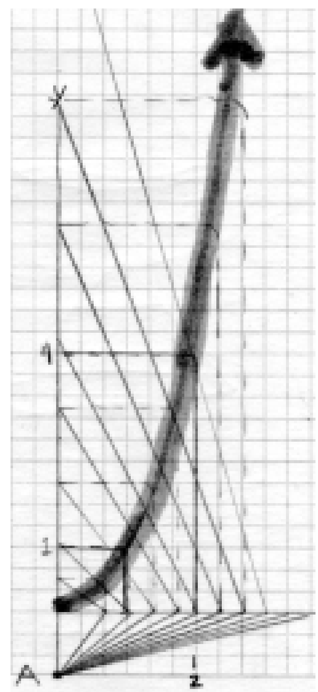


Image courtesy of Mike Vander Nat

Lines drawn from A to the horizontal axis make right-angle turns to intersect the vertical axis. The combination of the points on the two axes forms a parabola.

FIGURE 8

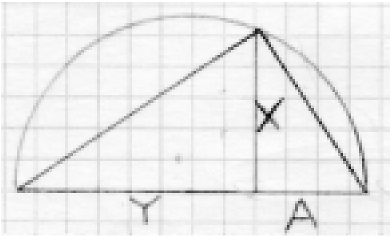


Image courtesy of Mike Vander Nat

FIGURE 9

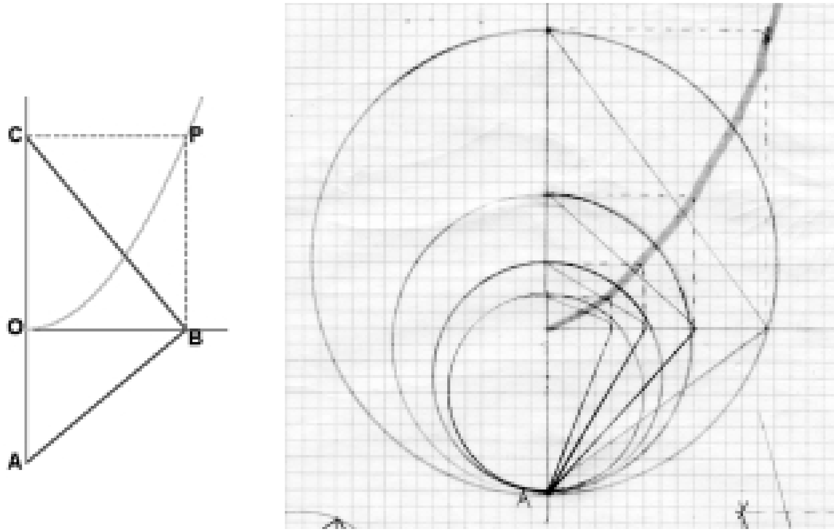


Image courtesy of Mike Vander Nat

Combining a number of these triangles creates the parabola

with the axis can be thought of as making two similar triangles, making the ratio $A/X = X/Y$ (**Figure 8**). We then get $AY/X = XY/Y$, and $AY/X = X$, which gives $AY = X^2$. So, when $A = 1$, $Y = X^2$ (**Figure 9**).

Each horizontal motion is "wedded" to a vertical change of squared relationship to the horizontal. Their union, the parabola, expresses the process of squaring.

But what if we take the entire complex field? This is a two-dimensional space, and each result of squaring is two-dimensional as well. Together, that makes four dimensions! No wonder d'Alembert, "rests not on geometric but on purely analytic principles "

Gauss resolved this with the logarithmic spiral. If each rotational doubling squares length, we could express any location $(a+b\sqrt{-1})$ as $r\,(\cos\varphi+\sqrt{-1}\,\sin\varphi)$ (**Figure 10**).

And, squaring it spirally, we get $r^2\,(\cos 2\varphi+\sqrt{-1}\,\sin 2\varphi)$.

Do you recognize anything from Gauss's 1799 paper? Gauss simply applies this transformation to his entire algebraic equation $X = x^m + Ax^{(m-1)} + Bx^{(m-2)} + \text{etc.} + Lx + M = 0$, creating for each x, $r(\cos\varphi+\sqrt{-1}\,\sin\varphi)$ instead, and producing:

$$(1)\quad r^m \cos m\varphi + Ar^{(m-1)}\cos(m-1)\varphi + Br^{(m-2)}\cos(m-2)\varphi + \ldots + Krr\cos 2\varphi + Lr\cos\varphi + M = 0,$$

and

$$(2)\quad r^m \sin m\varphi + Ar^{(m-1)}\sin(m-1)\varphi + Br^{(m-2)}\sin(m-2)\varphi + \ldots + Krr\sin 2\varphi + Lr\sin\varphi + M = 0.$$

This keeps separate the parts with and without $\sqrt{-1}$, *geometrically* constructing two surfaces, where d'Alembert only falsely ruminated on one, non-existent curve (**Figure 11**).

From these beginnings, Gauss is able, in his 1799 paper, to simply and elegantly use the ontologically transcendental geometric nature of number to demonstrate a characteristic (the fundamental theorem) of its shadow, algebra. How foolish are those who seek to explain the universe by imagining that its shadows are reality!

—*Jason Ross*

Notes

1. How much time, effort, and money is annually wasted by students attempting to explain "financial economics" from monetary theory? Perhaps they could put their time to good use by providing a thorough accounting of such waste, *per annum.*

2. Bruce Director, "Gauss's Declaration of Independence" and "Bringing the Invisible to the Surface," *Fidelio,* Fall 2002.

3. Carl Gauss, "The Metaphysics of Complex Numbers," translated from Gauss's *Werke*, Vol. 2, pp. 171-178, by Jonathan Tennenbaum in *21st Century Science & Technology,* Spring 1990.

See http://www.wlym.com and http://www.wlym.com/~jross/gauss/ for Gauss's referenced paper and work by the LYM on Gauss's 1799 paper.

FIGURE 10

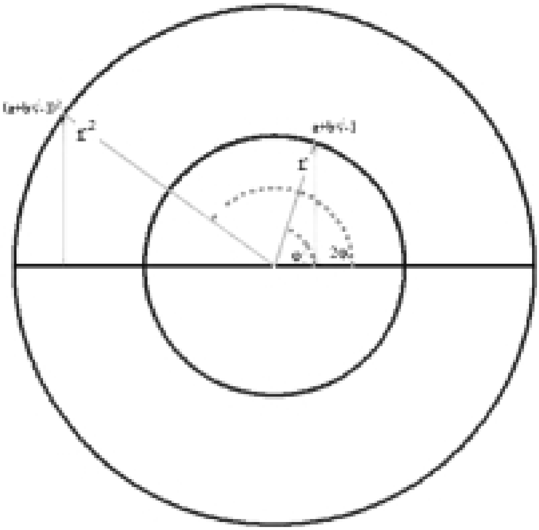


FIGURE 11

EIRNS/Dan Sturman

A geometric construction corresponding to Gauss's Fundamental Theorem of Algebra (right), created by the LYM in Philadelphia.