Box 19

The Catenary

"The resourcefulness of this curve is only equal to the simplicity of its construction, which makes it the primary one among all the transcendental curves."

—*G.W. Leibniz,* On the Catenary Curve, *1691*

Leibniz, knowing the order of the universe to be developing in accordance with perfection, by which the simplicity of its means carries out the richest accomplishments, sought to bring the state of mankind into coherence with the discoverable reality of such a universe.

The simplicity of its means shines forth in Leibniz's investigation of the catenary, a curve he defined as expressing "least action." This curve hangs the universe in perfect suspension amongst every infinitesimal point, and thus, most simply expresses the pathway of gravity's ordering of the material world. The catenary's productivity exceeds all other curves, in its power to generate all algebraic powers from itself, thus truly demonstrating the power to accomplish the richest effect.

The constantly changing nature best expresses Leibniz's calculus, in which all matter and motion is constantly guided, not through sense perception, or connecting dots and determining algebraic equations, but through a set of unseen relationships demanding themselves to be maintained throughout, as in a curve changing its pathway, thus pointing to an unseen physical principle existing universally throughout the curve. These principles, reflected as a guiding relationship, exist at even the smallest interval of change, as along the catenary, where least action is maintained even at the point the empiricists call nothing, or zero: the point at the exact bottom of the chain.

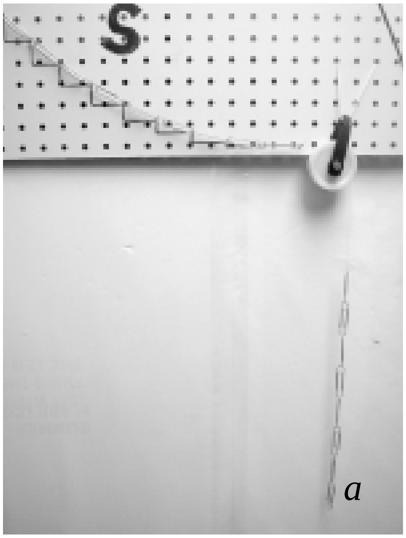
Thus Leibniz, leaving the world of changeless chaos of sense perception to the beasts, solved a seemingly unsolvable paradox of sense perception, in which a constantly changing universe, such as a pathway of constant curvature, can be known through paradoxical infinitesimally small points, which are the most simple, but also have the most power. Therefore, in discovering the reason for the catenary curve, opening up a whole new realm of science, Leibniz experimentally demonstrated to mankind that the universe is one of a perfect Creator, one designed for the human mind to discover its eternal truths. Even while he was often occupied with "responsibilities of a totally different nature," that is, launching a global political renaissance reaching the shores of North America and extending as far as China, Leibniz saw that improving the method by which humanity could discover principles and apply them to further increase the perfection and power of the human mind, results in profound developments for the human species as a whole, and thus is the only means to change the state of mankind. This is the power of the catenary.

Catenary Curvature

"The first to consider this curve, which is formed by a free-hanging string, or better, by a thin inelastic chain, was Galileo. He, however, did not fathom its nature; on the contrary, he asserted that it is a parabola, which it certainly is not. Joachim Jungius discovered that it is not a parabola, as Leibniz remarked, through calculation and his many experiments. However, he did not indicate the correct curve for the catenary. The solution to this important problem therefore remained for our time."

—*Johann Bernoulli,* Lectures on the Integral Calculus, *1691*

FIGURE 1



The catenary is the curve formed by a hanging chain, whose constantly non-constant curvature is acted on by the pull of gravity, and horizontal tension. Its changing vertical/horizontal relationship can only be determined physically, by these two forces, and cannot be expressed algebraically in any Cartesian coordinate system. Is the one power determining the interaction of these forces knowable?

Hang a chain between your hands. Keeping the chain in one place, have someone else pinch a lower portion of the chain. Let go of the extra chain! Does it change its structure? No. The total weight between your hands changes, but not the

FIGURE 2

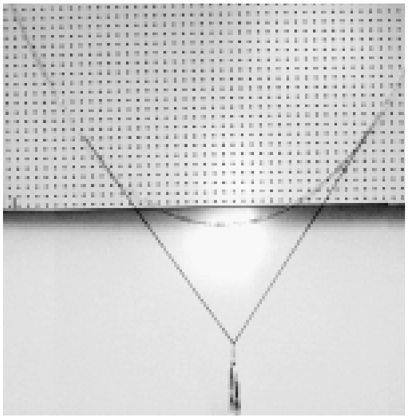


FIGURE 3

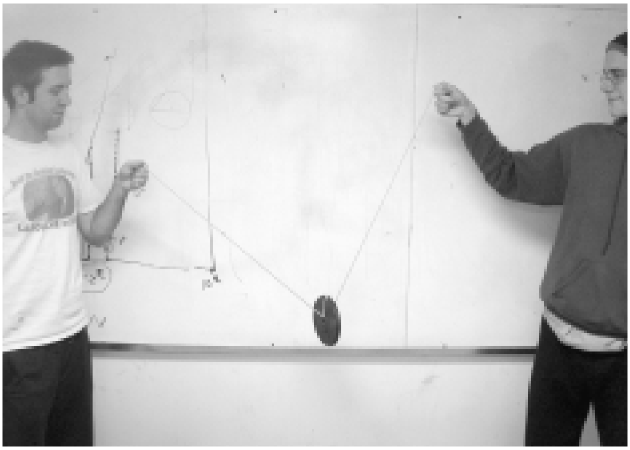
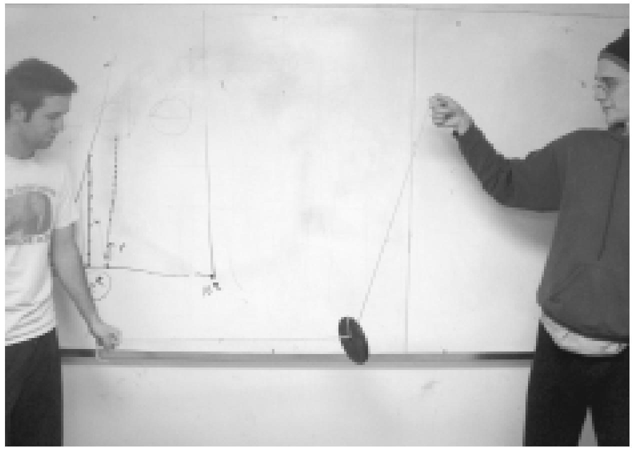


FIGURE 4



structure of the chain. Although the vertical force increases as the amount of chain increases, the horizontal force stays constant; this can be discovered by finding the horizontal force at the bottom of the catenary and observing the effect as you remove lengths of chain. Does the horizontal force change (**Figure 1**)?

The constant horizontal tension and the vertical force of gravity have an unseen, changing relationship as you change the position of your hands on the chain. To find out how these forces determine the curve, it is necessary to use more than the senses.

Therefore, proceeding to the unseen, remove a portion of chain and replace it with a weight hanging tangent to the curve. What do you observe? If your measurements are correct, the links holding up the weight, equal to the chain removed, do not move, nor do they notice the change. Therefore, because the weight of chain exerts its action at the tangent points and the pull of the weight is equal, whether you have the catenary or a proportionate amount of weight hanging at the intersection of the tangents, the unseen relation of vertical and horizontal force acting to determine the curvature of the chain, can now be discovered and measured precisely, using this method of tangents (**Figure 2**).

Now, hang a weight on a rope. If it is not swinging from side to side, it is clear that the horizontal tension is constant, while the vertical force on either part of the rope changes as the angles change (**Figure 3**).

Hold the weight still, and rotate one of the ropes perpendicular to the pull of gravity (**Figure 4**).

At this moment of the experiment, a singularity of the physical relationships arises: The force pulling on that end of the rope is horizontal only, with no vertical component. At only this singular point, the relationship between the constant horizontal force and the force of the vertical weight pulling down is found to correspond with the ratio of the sines of the two angles α and β, which correspond with the vertical and horizontal lengths X and Y (**Figure 5**).

Since the chain, or the weight hanging on the tangents, has an equal effect on the tangent links, the relation of the whole weight E to the horizontal force at B can similarly be expressed as the relation of the whole chain AB to the length of chain a shown in Figure 1, whose weight is equal to the horizontal force at bottom. Therefore, the vertical and horizontal change expressed as length X and length Y can be expressed in a proportionate relationship with length AB and a. $X/Y = AB/a$.

In other words, the relationship of forces is transformed back into the relationship corresponding to our original length of the catenary chain, and therefore, the physical forces are discovered to be proportional to the vertical/horizontal change.

But, is this relationship constant throughout the chain? Using the method of Leibniz's calculus, an infinitesimally small change of the tangent will result in an infinitesimally small change in X and Y, expressing the same relationship. Therefore, the relationship of the two forces is precisely proportionate to the change of X and Y at every point; in other words, an infinitesimally small point expresses the relationship guiding the whole curve.

The unseen physical characteristic is brought into view by way of a single "point." This point does what no other

FIGURE 5

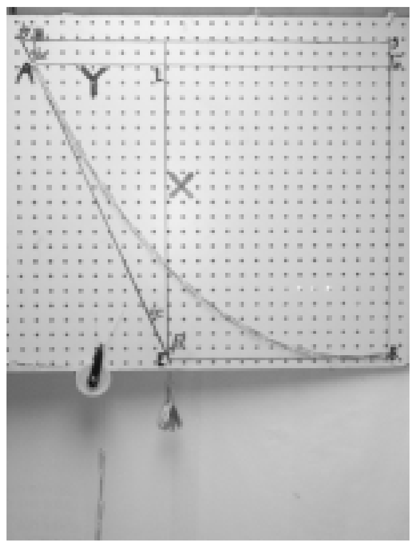
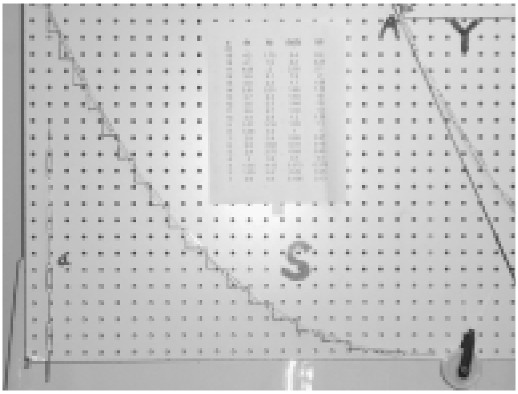


FIGURE 6



s	dx	dy	dx/dy	s/a
20	-		-	
19	4.2	1.75	2.4	2.4
18	4.1	1.9	2.2	2.25
17	4.09	2	2.05	2.1
16	3.9	2.1	1.9	2
15	3.8	2.2	1.7	1.88
14	3.81	2.31	1.65	1.75
13	3.7	2.4	1.54	1.6
12	3.6	2.5	1.44	1.5
11	3.6	2.7	1.33	1.3
10	3.4	2.8	1.2	1.25
9	3.27	3.16	1.03	1.1
8	3.08	3.2	1	1
7	2.9	3.4	0.85	0.87
6	2.6	3.62	0.72	0.75
5	2.4	3.75	0.64	0.63
4	2	3.9	0.5	0.5
3	1.55	4.15	0.373	0.375
2	1.25	4.2	0.29	0.25
1	0.6	4.4	0.136	0.13

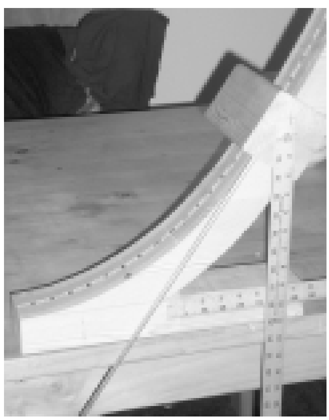
point on the catenary does. Acting as a true singularity of *physical* geometry, it most clearly expresses the unseen physical power ordering the curvature of the catenary.

Figure 6 demonstrates this discovery for 20 points of tangency, where S is taken as different lengths of the catenary and a is the constant equal to 8 paperclips. Here, in looking at the data, observe the relationship that exists even while the parameters are changing constantly. Hypothesize what relationship is demanding itself to be maintained, although showing up as changing in each differential expression. Can this be known in any other way but through its physical relationship?

To animate this new idea even further, examining these physical forces solely as changing lengths, a proof of the principle using a machine tool was constructed to continuously demonstrate the differential expression $S/a = dx/dy$. The measurements taken are shown in **Figure 7.**

FIGURE 7

a	S	S/a	X	Y	X/Y	Diff.
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]



Natural-Logarithmic Function

To repeat what was said above, the catenary curve cannot be known from any algebraic function. Leibniz, seeking for a “type of expression, as well as the best of all possible constructions, for transcendentals” was led toward a “higher domain for which new avenues needed to be opened.” He found the catenary to be constructible as the arithmetic mean between two logarithmic curves, one constructed inversely to the other. Thus the catenary is a function of *two* non-algebraic functions (**Figure 8**).

What physical construction are these two inverse logarithmic curves derived from?

Try a doubled cone of 90° cut perpendicular to the base. This creates a hyperbola (**Figure 9**).

Looking back at Bernoulli’s lectures on integration, one sees that he demonstrates that the hyperbola grows in area arithmetically, while the lengths grow geometrically. Hence, he constructs the essence of the equilateral hyperbola: the natural logarithmic curve, a curve of arithmetic growth in one direction and geometric growth in the other, with a subtangent of 1 (**Figure 10**).

Now, return to the double cone and construct a logarithmic curve from the curves of the hyperbola on either side. Are these the two curves that Leibniz uses to construct the catenary? How can we replicate his construction with our two invisible logarithmic curves on opposite sides of the cone? What is required to bring these curves into an inverse relationship (**Figure 11**)?

To construct the relationship of the natural logarithmic curves that Leibniz designed, one curve must swing around the zero point on the axis, i.e., the vertex of the double cone. By what amount? An

FIGURE 8

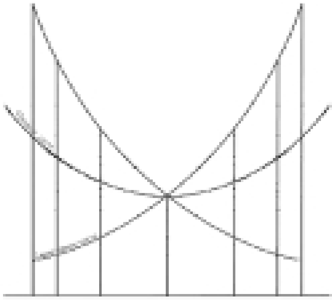


FIGURE 9

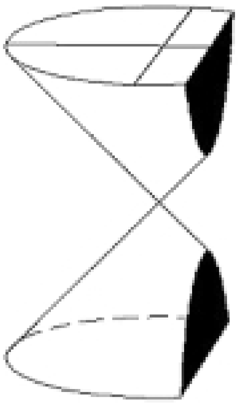
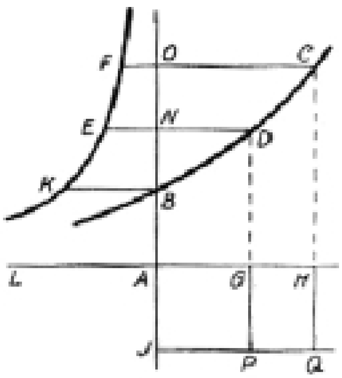


FIGURE 10



"imaginary" one! (**Figure 12**)

Thus is found Leibniz's construction, in a new domain, existing paradoxically from the standpoint of the sense-perceived cone. As Leibniz proclaimed: "[T]he Divine Spirit found a sublime outlet in that wonder of analysis, that portent of the ideal world, the amphibian between being and not being, which we call the imaginary root of negative unity."

How did Leibniz discover this?

FIGURE 11

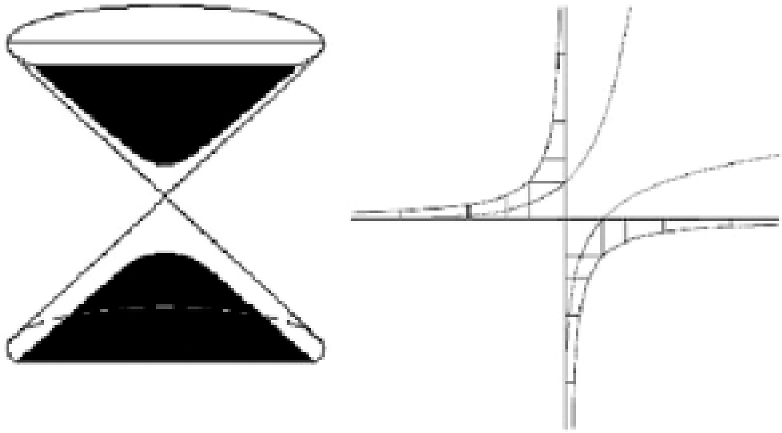
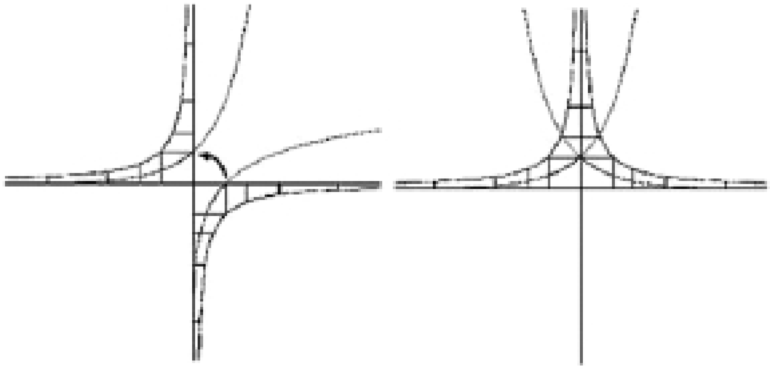


FIGURE 12



Investigate this construction more closely. What is the geometric mean between the two logarithmic functions? Well, the height of the logarithmic curve below the catenary is to the height of one, as one is to the height of the logarithmic curve above the catenary. In other words, the geometric mean is the tangent to the *point* at the bottom of the catenary, which is, ironically, the point betraying the unseen *physical* power generating the curvature of the catenary (**Figure 8**).

"Even though my hands were tied," Leibniz wrote in 1691, "and I could not busy myself with this as I should have, there was a higher domain for which new avenues needed to be opened; so, this is what was important in my eyes: That is, the case of developing methods is always more crucial, than particular problems, although it is the latter which usually bring applause."

—*Michael Kirsch and Aaron Yule*