Box 2

Constructing Volumes

Construct!

Members of the Seattle LYM work on the problem of constructing various volumes.

The difference between a real economy:

Here LYM members contemplate the magnificent construction of the Grand Coulee Dam.

and the fantasy of a financial analyst:

Here, we see human activity wasted on the "virtual economy," known as the stock exchange.

is construction. Construction tests the viability of those ideas the mind thinks best conceived: Are they really of legitimate parentage, or did an adulterer slip in when your guard was down, and adulterate the whole affair?

You may think: "Ah, I know this! This is simple. . . ." But when you try to pull your idea from your mind into the visible world . . . well, it was not nearly so simple as you thought! The mind rushes, unencumbered by the material world, capable of conceiving of perfectly consistent systems, glorious designs, elaborate . . . machinations . . . which have little relation to reality. The body, meanwhile, weighed by its own flesh, mucks in the mud, capable of pursuing little but the sensual pleasure of a pig. Where is their connection?

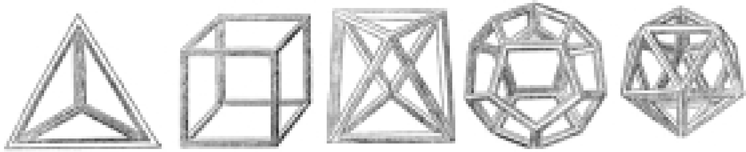
Construction is the mean between mind and body; it is the means of making music through a harmony of these two diametrically opposed elements. It is the only means of investigating reality. If you take up the challenge laid out here by Lyndon LaRouche, if you get your hands dirty in pursuit of its solution, you were likely to produce an idea directly related to the idea which determines what I am now writing, as I attempt to convey the fruits of our struggle with LaRouche's challenge. You were likely to laugh, as we did—and as I suspect LaRouche did—when he wrote out the problem as he did. In just a few words, he presents an inquiry which takes many hours, and really, many people, to adequately investigate. And if that were not enough, there is an element of the seemingly impossible which we were immediately aware was embedded there.

First, LaRouche asks us to think of

the volume of water a cube could contain "as compared with the relevant sphere or torus of the same capacity." If he means what he says, he asks us for a "cubature of the sphere": He asks us to produce a cubical volume equal to the volume of the sphere. This is certainly no less a problem than the quadrature of the circle, and actually, a good deal more of a problem.

The quadrature of the circle is the process of making ever-closer approximations of the length of the perimeter of the circle by drawing circumscribing and inscribing polygons of an ever-increasing number of sides, as Archimedes did. The process is intended to result in the creation of a square whose area is exactly equal in length to the area of the circle. Archimedes applied to the circle a method associated with Eudoxos, a friend of Plato, called "exhaustion." The method of exhaustion had worked well to produce precise results for other problems, like the quadrature of the parabola, and it was likely used with similar effect on some of the volumetric problems we encounter below.

FIGURE 1



These Platonic Solids, drawn by Leonardo da Vinci, are the only regular solids possible to construct within a sphere. They point to one crucial difference between surfaces and volumes. (Try bisecting the sides of the octahedron, to make a solid with 16 faces, the way you would bisect the sides of the octagon to make a polygon with 16 sides, to fully understand what I mean.) Also note that because of its "regularity," its equal-sidedness, the cube is "spherical." (We will see more on this in a moment.)

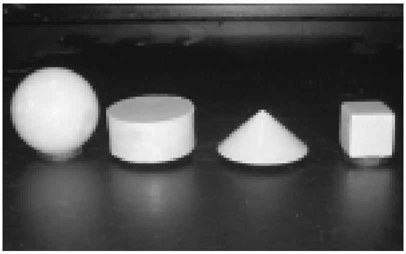
But Nicholas of Cusa showed that a true quadrature of the circle is ultimately impossible because of the "species difference" between the curved line of the circle and the straight lines of the polygons, as discussed in Box 1. The cubature of the sphere is certainly related to this problem, but while the number of polygons that can be inscribed in a circle is infinite, there is a limited number of solids that can be inscribed in the sphere (**Figure 1**).

LaRouche then calls for a cylinder and cone "each able either to contain that amount of water, or to double that amount in the cylinder." This requires determining the relations among cube, sphere, torus, cylinder, and cone (**Figure 2**). Perhaps you, like some of us, were trained in school and can spout out the formulae for the volume of the sphere, cylinder, and cone as a Pavlovian response. Perhaps, you were not able to contain yourself, even as the problem was first posed. If this is so, you must find an incredulous person, or better yet, muster incredulity yourself, and consider this paradox: We are told that the volume of the cone is less than one half the volume of the cylinder (**Figure 3**). (The fun is figuring out how much less.)

But, as the incredulous person will

Box 2 continues on next page

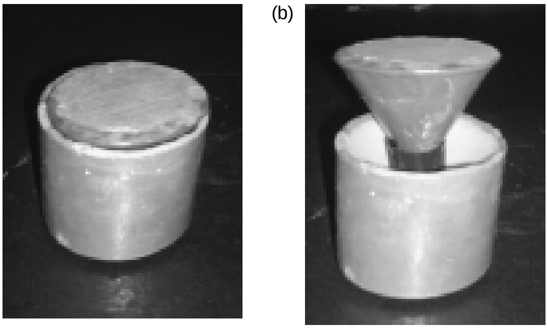
FIGURE 2



The side of the cube is equal to the radius and height of both the cone and cylinder, and to the radius of the sphere. (We apologize for the glaring absence of the torus.)

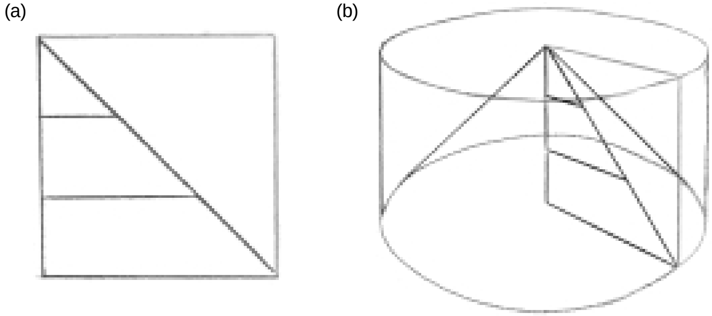
FIGURE 3

(a) (b)



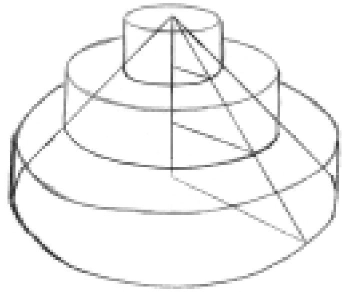
A cylinder (a) and the cone that fits into it (b). The cone has the same base and height as the cylinder.

FIGURE 4



If you rotate the rectangle (a) around its left edge, you will produce the cylinder (b). If you rotate the right triangle formed by cutting the rectangle in (a) along its diagonal around the same edge, you will produce a cone that has the same base and height as the cylinder, as seen in (b).

FIGURE 5



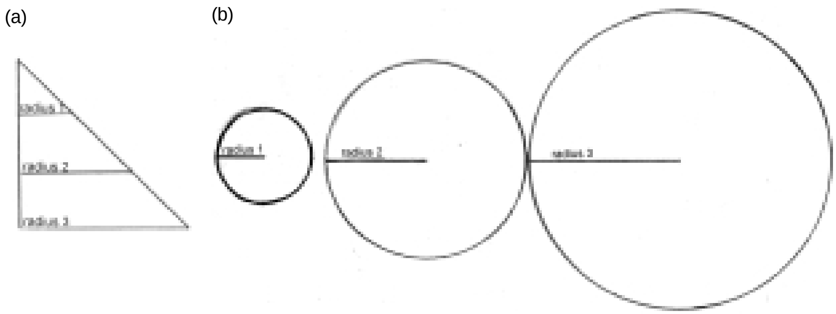
The height of each cylindrical layer is 1/3 the original height of the cone. The base of each cylindrical layer has a radius equal to the base of triangle produced by that cut. The first, smallest base has a radius 1/3 the radius of the cone; the next base has a radius 2/3 the radius of the cone; and the final base has a radius equal to that of the cone.

point out, the cylinder can be produced as a volume of rotation, the effect of rotating a rectangle about an axis that coincides with its edge. If you cut that rectangle in half along its diagonal, you will have a right triangle which is half the area of the original rectangle (**Figure 4**).

Given this fact, "reason" leads to the conclusion that the volume of the cone will be exactly half that of the cylinder. Of course the reason used here, is none other than the "lazy reason" that Socrates spurns in the *Phaedo,* or the sloppiness Eratosthenes ridicules in the playwright who has a character proclaim that the tomb of a king is too small, and therefore the tomb should be doubled, by doubling the length of each side. Clearly, Eratosthenes tells us, this is a terrible blunder, for the volume would now be eight times greater, which the playwright could have known, if he only took the time to think about it.

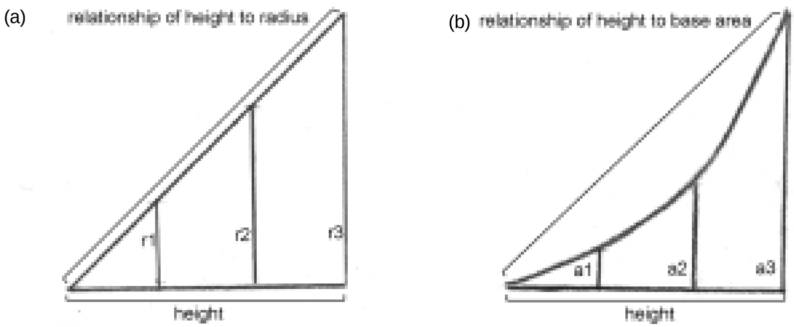
Now consider the cone: Think of it as a series of cylinders added up together; this is akin to Eudoxus' method of exhaustion mentioned above (**Figure 5**). The radii of the series of diminishing cylinders changes in arithmetic proportion relative to the number of cylinders chosen, but the areas of their bases, and hence their volumes, would change as the square of that radius (**Figure 6**). The cone's volume changes in a non-arithmetic way, mak-

FIGURE 6



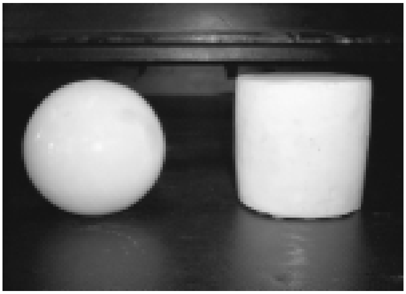
The three radii in (a) correspond to the three areas shown in (b).

FIGURE 7



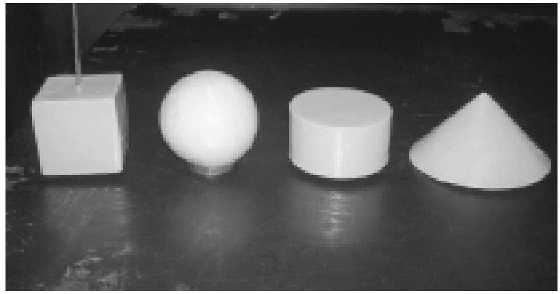
A graphical representation of the essential difference between the volume of a cone and cylinder. The vertical lines in (a) represent the various radii. The vertical lines in (b) are equal to the corresponding squares of those radii.

FIGURE 8



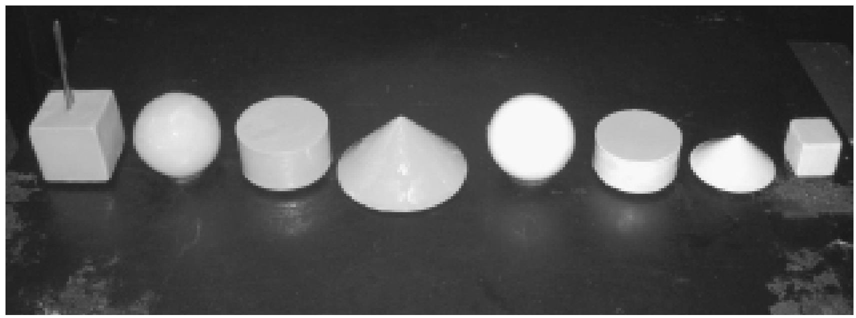
Here we have a cylinder, the base of which has a radius equal to the radius of the sphere, and the height of which is equal to the diameter of the sphere.

FIGURE 9



These solids all have the same volume, as determined from the volume of the sphere. (Again, forgive the absence of the torus.) Ask yourself, how did we determine these volumes? Each posed a particular problem of finding a cube root. Finding the volume of the cube was nearly impossible!

FIGURE 10



The four solids on the left are of equal volume. The original solids are on the right. In the original set of solids, the cylinder and cone both have a radius and height equal to the radius of the sphere, and the side of the cube is equal to the radius of the sphere. Notice the dramatic difference in the size of the two cubes and the two cones. The two spheres are the same size.

ing the relationship between the volume of rotation of the triangle and rectangle, between the cone and cylinder, different than the relationship between the areas of the triangle and rectangle (**Figure 7**). This is another difference between the surfaces and solids, with which we must grapple.

The relationship between the cylinder and sphere can be adduced in a similar way. First build a cylinder with a radius equal to that of the sphere, and a height equal to that of the sphere's diameter (**Figure 8**). Then weigh them (note that this works only if they were made of the same material), and compare their weights. Ask, why is this true? Why did we get this result? This provides additional insight into the problem.

But then you are reminded, as if remembering something nearly forgotten, we must now construct a sphere, torus, cone, cylinder, and cube with the same volume! Although related to the preceding exploration, this adds a new element to worry us (**Figures 9 and 10**).

Now we come to the question of doubling these volumes, and the geometric effect in this doubling. There are three ways in which the volume of a rectangular solid can be doubled (**Figure 11**). This is also true of the cylinder and cone (**Figure 12**). In the images shown in **Figure 13**, only one of the three doubled volumes is similar to the first.

In like manner, the sphere can only be doubled in one way, because a sphere must always be similar to any other sphere. (Ponder the implications of this for a moment.) The cube must be similar

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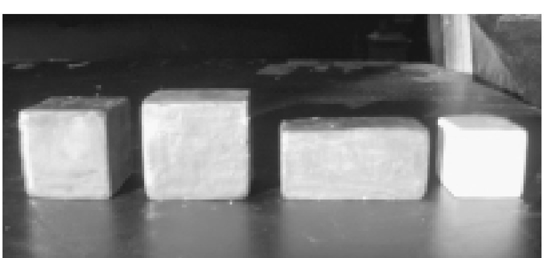
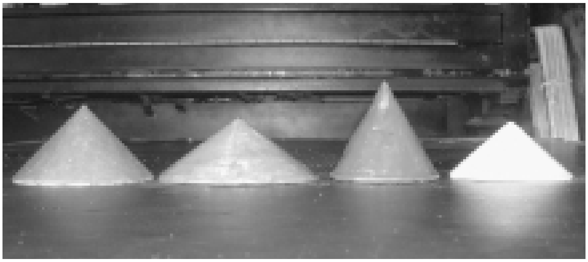
FIGURE 11



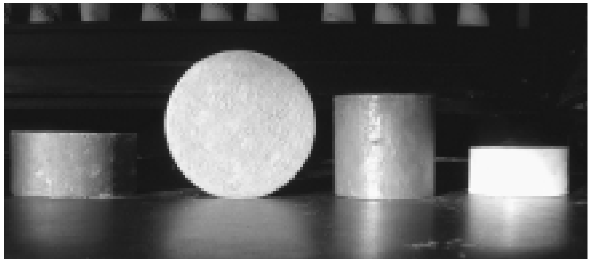
Our original cube, whose side is equal to the radius of our sphere, is at the far right. Next to it is a rectangular solid whose width is double that of the cube, while its height and depth are the same as the cube. The third solid to the left has a face that is double the face of the original cube, but its depth is the same as the cube. Both of these solids are double the volume of the original cube, and their construction did not require that we find a cube root. But the fourth solid on the left is the doubled cube. Its construction required a profound addition to our array of capabilities.

FIGURE 12

(a)

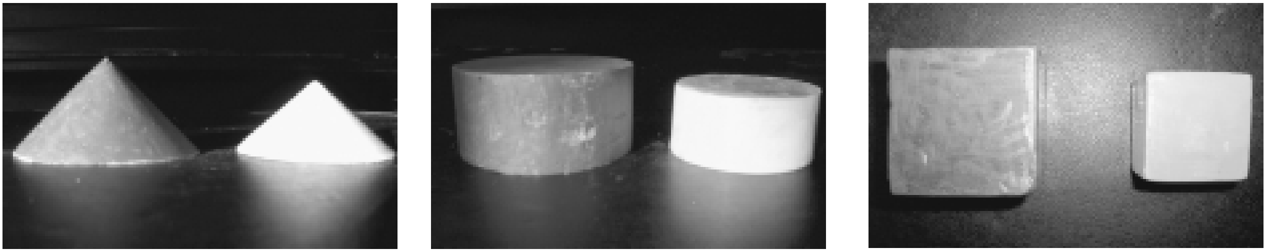


(b)



In both (a) and (b), the original volume is on the far right, and the perfectly doubled similar volume is on the far left. In (a), each of the three cones next to the original cone is double the volume of the original. The first to the left is doubled by doubling the height, the second by doubling the area of the base. The cone on the far left was doubled by an equal increase to both the radius of its base and its height, producing a similar cone. In (b), we show the same results for the cylinder. The base of the cylinder third from the right (shown on edge) is doubled.

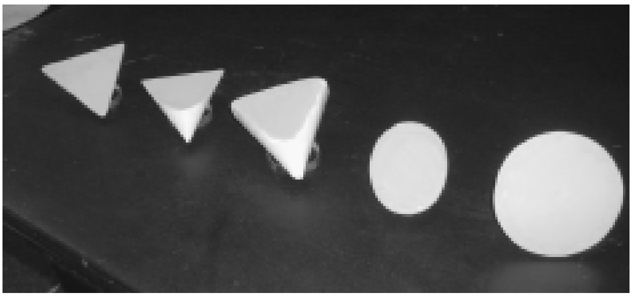
FIGURE 13



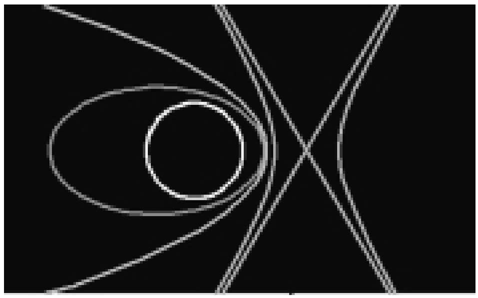
Here we show each original solid with its similar companion of double capacity. Because of the difficulty posed by constructing hollow containers, we realized that if our solids were constructed properly, we could make use of a discovery of Archimedes to determine their volumes.

FIGURE 14

(a)



(b)



In (a), we show the various conic sections progressing from the horizontal cut, which gives the circle on the far right; to a cut less than parallel with the side of the cone, which results in an ellipse; to the cut parallel with the side, which gives the parabola; to a cut between the angle of the side and vertical, which gives the hyperbola. The final cut shown is that made down the axis of rotation, which reveals the triangle rotated to produce the cone. In (b), we show a schematic produced by Bruce Director to demonstrate Kepler's conception of the conic functions. As the focus moves off to the left, the circle is transformed into an ellipse. At the boundary with the infinite, the ellipse becomes a parabola. The hyperbola is formed on the "other side" of the infinite.

to any other cube, so in this way, it is a spherical solid. Look back at the problem of constructing volumes of equal capacity.

There are ways of cheating in constructing a cone or cylinder whose volume is equal to that of a sphere. If you are unconcerned that the solids you produce are similar to your original objects, the problem is as easy as changing the height, or the surface area of the base, of the original. But then you miss the fun of confronting the construction of *a series of different* cube roots. Even if you try to avoid this difficulty, you can not escape the problem of finding a cube root (and a very strange cube root at that), when constructing a cube with equal capacity to the sphere.

In this experiment with volumes, which is at heart a study of cubes, the problem of the curved and the straight lurks around every corner (and around every edge). When Kepler spoke, in his *Optics,* about the relationship among the conic functions, looking at the different conic sections as a continuous transformation from the perfectly curved, the circle, to the perfectly straight, the straight line, he was, in truth, depicting the aspects of curved and straight married in the cone itself (**Figure 14**).

In this regard, the cone and cylinder obviously share this important characteristic, this union of curved and straight, as seen in their sections (**Figure 15**).

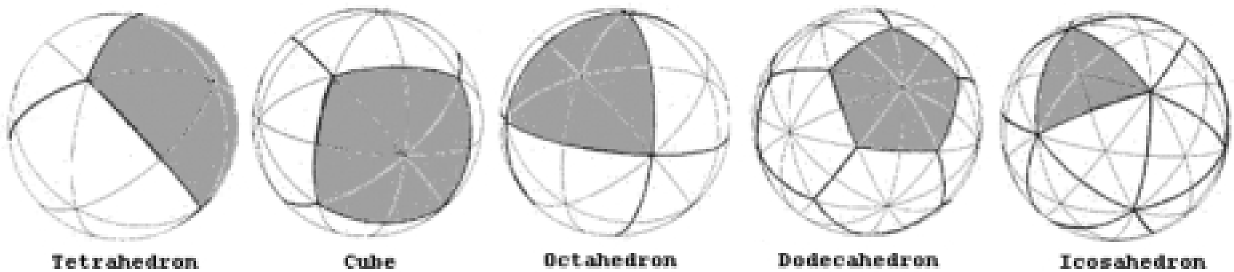
But the cube, which does not appear to have any part of curvature within it, is itself spherical! (**Figure 16**)

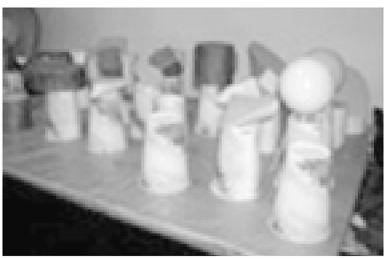
To conclude, consider the torus, so neglected in this initial treatment. Where does it belong? And, how do you construct those cube roots, anyway?

FIGURE 15

Here we show that there are only three different cuts of the cylinder, no matter how you cut it! (The axial cut that produces a rectangle is not shown.) Notice that the cylinder and cone share the circular and elliptical cuts (although in the cylinder all its elliptical cuts are of a special type), but that the parabola and hyperbola are unique to the cone.

FIGURE 16





—The entire Seattle LaRouche Youth Movement was involved in this project. In addition to Niko Paulson, Peter Martinson, and Riana St. Classis, Dana Carsrud, and Will Mederski consistently aided the project's progression to this stage of completion. They helped construct the means of constructing the solids, and helped construct the solids, paint them, epoxy them, and photograph them. And now, we shall all play with them! Photographs were taken by Lora Gerlach, Will Mederski, Dana Carsrud, and Riana St. Classis. Lora Gerlach also provided priceless assistance with navigating the digital flat lands of Photoshop and Word.