Box 7

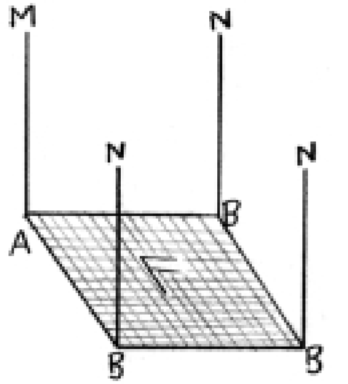
Gauss, Bolyai, and Anti-Euclidean Geometry

> *“I would also note that I have in the last days received a small paper from Hungary on Non-Euclidean geometry, wherein I find reflected all of my own ideas and results, developed with great elegance—although for someone to whom the subject is unknown, in a form somewhat hard to follow, because of the density. The author is a very young Austrian officer, the son of a friend of my youth, with whom I discussed this theme very much in 1798, although then my ideas were much further from the development and maturity, that they have attained through this young man’s own reflection. I hold this young geometer v. Bolyai for a genius of the first order.”*
>
> *—Gauss to Gerling, Göttingen, Feb. 14, 1832*

János Bolyai’s book, *The Science Absolute of Space*, billed itself as “exhibiting the absolutely true science of space, independent of the eleventh axiom of Euclid, (which cannot be decided *a priori*), with the geometrical quadrature of a circle in the case of its falsity.” His method of investigation was the following:

Take all lines *BN* parallel to a given line *AM*, and perpendicular to the line connecting their endpoints *B* and *A*, and the complex of such points *B* will form a surface, *F* (**Figure 1**).

FIGURE 1



Transform plane *F* such that all *BN* cut *AM* in *A:* Now, rather than maintaining the assumption that parallel lines never intersect, let us assume, instead, that they do (and, as Bolyai proves, necessarily in the same point *N*). Our surface *F* becomes something different (**Figure 2**):

Bolyai then proves “. . . [I]t is evident that Euclid’s Axiom XI and all things which are claimed in geometry and plane trigonometry hold good absolutely in *F, L* lines being substituted in place of straights: therefore the trigonometric functions are taken here in the sense as in Σ. . . .”

But, he demonstrates that several par-

FIGURE 2

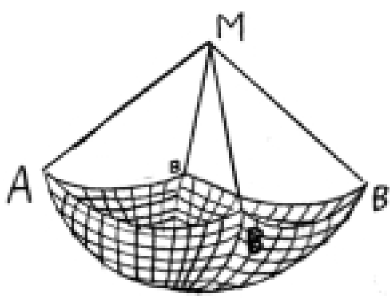
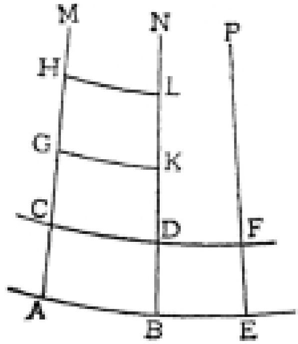


FIGURE 3



adoxical things become possible, such as that there are cases where the lines of area *AMEP,* although larger than *AMBN,* can be moved, without stretching, to fit exactly over the lines of the latter (**Figure 3**).

With Euclid, there can be no such mapping; however, Bolyai has shown this to be possible even with a congruency of *AMEP* with *AMBN,* resulting out of parallel lines “. . . which is indeed singular, but evidently does not prove the absurdity of *S* [*S* = Bolyai’s geometry].”

Abraham Kästner’s task of constructing a geometry free of the parallel postulate, however, had remained unfulfilled by Bolyai’s work. Gauss, although impressed by the work of this young man—which exhibited results that he had obtained many years prior, but never published—recognized that Bolyai, although attempting to undertake a revolutionary investigation into the nature of physical space, neglected to investigate the nature of the tools used in that investigation. The fundamental questions concerning the actual, ontological, existence of straight lines and curves were never questioned, but rather treated as playthings handed down by God.

Gauss’s Letters on Anti-Euclidean Geometry

The following letters were Gauss’s method of working his contemporaries through the difference between a Non-

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Euclidean geometry as a mere mathematical model, and Anti-Euclidean geometry as the only truly physical geometry. (These letters can be found in *F. Gauss, Werke,* Band 8 (Göttingen, 1900.)

"All my efforts to find some contradiction, some inconsequence in this Anti-Euclidean geometry have been fruitless, and only one thing therein resists our understanding; that is, that, were it [Anti-Euclidean geometry] true, there must be in space some linear magnitude (though to us unknown), determined in and of itself. However I suspect, in spite of the meaningless word-wisdom of the metaphysician, that we actually know too little or nothing at all about the true nature of space, as to be allowed to mix up that which seems unnatural to us, with the absolutely impossible. Were the Anti-Euclidean geometry the true one, and the above mentioned constant in a reasonable relation to such magnitudes which lie within the domain of our measurements on the Earth or lie in the sky, one could ascertain them *a posteriori*."

—Gauss to Taurinus, Nov. 8, 1824

"Anti-Euclidean geometry contains nothing contradictory, although some people at first will consider many of its results paradoxical—the which, however, to consider as contradictory, would be a self-deception, arising from an early habituation to thinking of Euclidean geometry as rigorously true. . . . There is nothing contradictory in this, as long as finite man doesn't presume to want to regard something infinite, as given and capable of being comprehended by his habitual way of viewing things."

—Gauss to Schumacher, July 12, 1831

"In order to treat geometry properly from the beginning, it is indispensable, to prove the possibility of a flat plane; the usual definition contains too much, and actually implies surreptitiously a Theorem already. One must wonder, that all authors from Euclid until most recent times worked so neglectfully:

János Bolyai (1802-1860)

Alone this difficulty is definitely of different nature than the difficulty of deciding between Σ [Euclidean geometry] and S [Bolyai's Non-Euclidean geometry], and the former is not hard to resolve."

—Gauss to Farkas Bolyai, March 6, 1832

"Yet another subject which I have been thinking on during my scant free time, which for me is already almost forty years old, [is] the first foundations of geometry. . . . Here also have I consolidated quite a lot, and my conviction that we cannot fully lay the foundations of geometry *a priori,* has, where possible, become even firmer. Meanwhile I shall probably not come to publishing my very extended investigations for a long time, and perhaps this shall never occur during my lifetime, as I am fearful of the screeching of the Böetians, were I fully to speak out on my views. However it is curious, that apart from the known gap in Euclid's geometry—to fill which all efforts till now have been in vain, and which will never be filled—there exists another shortcoming, which to my knowledge no one thus far has criticized and which (though possible) is by no means easily remedied. This is the definition of a plane as a surface which wholly contains the line joining any two points. This definition contains more than is necessary to the determination of the surface, and tacitly involves a theorem which must first be proved."

—Gauss to Bessel, Jan. 27, 1829

"My purpose had been, as regards my own work, of which there is yet little on paper, to let nothing of it be known during my lifetime. Most people have no correct sense at all, as to what the crux of this matter is, and I have found only few people, who have taken up that which I have shown them, with any particular interest. In order to do that, one must have first rightly felt what is actually missing, and most people are totally unclear on this. Rather it was my intention, to bring everything to paper over time, so that it would at least not go under with me."

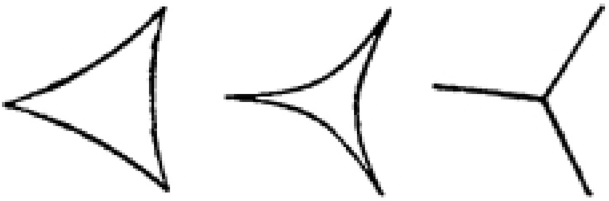
—Gauss to Farkas Bolyai, March 6, 1832

". . . [T]he path which I have taken, does not lead so much to the desired end, which you assure me you have reached, as to the questioning of the truth of geometry. Although I have found much which many would allow as a proof, but which in my view proves nothing (for instance, if it could be shown that a rectilinear triangle is possible, whose area is greater than that of any given surface), and therefore I am in a position to prove the whole of geometry with full rigorousness. Now most people, no doubt, would grant this as an axiom, but not I; it is conceivable, however distant apart the three vertices of the triangle might be chosen, that its area would yet always be below a certain limit. I have found several other such theorems, but none of them satisfies me."

—Gauss to Bolyai, Dec. 16, 1799

"It is easy to prove that, if Euclid's geometry is not the true one, there are no similar figures whatsoever: The angles in an equilateral triangle are also different as regards the length of the sides, about which I find nothing absurd. Then the angle is a function of the side and the side a function of the angle—naturally such a function, which at the same time contains a fixed line. It seems somewhat paradoxical, that a fixed line could simultaneously be possible *a priori;* I

FIGURE 4



however find nothing contradictory in that. It is even to be desired, that the geometry of Euclid not be the true one, as we would then have *a priori* a general measure, e.g., one could take as a unit of space the side of that equilateral triangle, whose angle = 59°59′59″.99999.” (**Figure 4**)

—*Gauss to Gerling, April 11, 1816*

Riemann’s Crucial Contribution

In 1854, the year before Gauss’s death, it would be his student Bernhard Riemann who, in presenting his habilitation dissertation, would lay the “Hypotheses Which Lie at the Foundations of Geometry” and finally fulfill Kästner’s request for a truly Anti-Euclidean geometry:

“If one premise that bodies exist independently of position, then the measure of curvature is everywhere constant; then from astronomical measurements it follows that it cannot differ from zero; at any rate, its reciprocal value would have to be a surface in comparison with which the region accessible to our telescopes would vanish. If, however, bodies have no such non-dependence upon position, then one cannot conclude to relations of measure in the indefinitely small from those in the large. In that case, the curvature can have at every point arbitrary values in three directions, provided only the total curvature of every metric portion of space be not appreciably different from zero. . . . Now however, the empirical notions on which spatial measurements are based appear to lose their validity when applied to the indefinitely small, namely the concept of a fixed body and that of a light-ray; accordingly, it is entirely conceivable that in the indefinitely small the spatial relations of size are not in accord with the postulates of geometry, and one would indeed be forced to this assumption as soon as it would permit a simpler explanation of the phenomena.

“The question of the validity of the postulates of geometry in the indefinitely small is involved in the question concerning the ultimate basis of relations of size in space. In connection with this question, which may well be assigned to the philosophy of space, the above remark is applicable, namely, that while in a discrete manifold the principle of metric relations is implicit in the notion of this manifold, it must come from somewhere else in the case of a continuous manifold. Either then, the actual things forming the groundwork of a space must constitute a discrete manifold, or else the basis of metric relations must be sought for outside that actuality, in colligating forces that operate upon it.

“A decision upon these questions can be found only by starting from the structure of phenomena that has been approved in experience hitherto, for which Newton laid the foundation, and by modifying this structure gradually under the compulsion of facts which it cannot explain. Such investigations as start out, like this present one, from general notions, can promote only the purpose that this task shall not be hindered by too restricted conceptions, and that progress in perceiving the connection of things shall not be obstructed by the prejudices of tradition. This path leads out into the domain of another science, into the realm of physics, into which the nature of this present occasion forbids us to penetrate.”

—*Sky Shields*
and Daniel Grasenack-Tente