Box 6

Kästner's Argument for Anti-Euclidean Geometry

"If two straight lines, in the same plane, are perpendicular to a third line, then they never intersect. This conclusion flows from the *clear* concept of straight line: for, on one side of the third line everything is identical to the other side, and so the two lines would have to intersect on the other side also, if they intersect on this side. But they cannot intersect twice. . . .

"However, when only one of the two lines is perpendicular to the third, and the other does *not* form a right angle, then do they intersect? And on which side of the third line? . . .

"Why should something *necessarily* occur with an oblique *straight* line, which does not *have* to occur, when one replaces it with a curved line? . . . Thus, the difficulty concerns the distinction between *curved* and *straight* lines. A curved line means, a line in which no part is straight. This concept of a curved line is *distinct,* because the concept of straight line is *clear;* but it is also *incomplete,* because the concept of straight line is *merely* clear."[1]

Well, to understand that, you'll have to understand this important parable: An information sciences student at MIT once fell in love with one of his classmates. He watched her every day, all day, as she went about her classes and other work, as she ate her lunch, and chatted with her friends; and so enamored was he that he finally rushed home one day, locked himself in his room and entered all of his observational data into his computer, creating the perfect replica, which he could keep on his desk. He proposed to it, it refused the offer, and he promptly threw himself out of the window into the traffic below. The young woman, who, unlike her *doppelganger,* had in reality been equally enamored with him, was not at all depressed, as she had already accepted the marriage proposal of the program she had written as a substitute for him.

Wellington: That's a bizarre story. What's your point?

George: The moral of the story is, that you can't mistake your image for the reality you tried to replace with it, no matter how much it seems to fit the facts. This was Abraham Kästner's point regarding Euclid's *Elements.* Every statement contained in it, individually, was the result of a truthful investigation undertaken by the greatest minds of the Pythagorean tradition, but the structure these truths were placed into by Euclid is false, on the face of it and, as a result, leaves us with shaky foundations, to say the least. For instance, is it true that the angles in all triangles add up to two right angles?

Wellington: Well, yes. If we call our triangle *ABC* (**Figure 1**), and extend sides *AC, CB,* and *AB* into *HD, CF,* and *AI,* respectively, and then simply add the line *GE* parallel to *HD,* we can say that the following things are true:

FIGURE 1

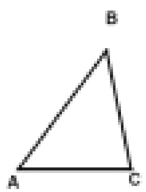
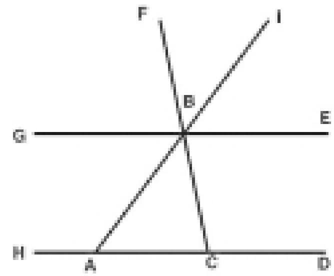


FIGURE 2



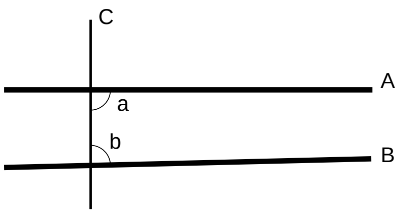
Angle *ACB* added to angle *BCD* gives two right angles, as can be seen immediately from the drawing (**Figure 2**), just as, if you turn the paper a little, you can see that angle *FBE* added to *CBE* gives two right angles. But, because lines *GE* and *HD* are parallel, angle *FBE* is equal to angle *BCD,* as can be seen. Therefore, angle *FBE* added to angle *ACB* must equal two right angles, the same as angle *FBE* added to *CBE,* making *ACB* and *CBE* equal. And since, again, angle *HAB* and angle *CAB* together make two right angles, and again, because line *GE* is parallel to line *HD,* angles *GBI* and *HAB* are equal. Therefore, angle *GBI* added to angle *CAB* gives the same thing as angle *GBI* added to angle *ABG,* so angles *CAB* and *ABG* must be equal. But angles *ABG, CBE,* and *ABC* together make two right angles, as you can see in the picture; therefore, angles *CAB, ACB,* and *ABC,* the three angles of the triangle, are equal to two right angles. And, if you followed that, you'll see that this can easily be shown for every triangle. That's proposition 32 in Book I of Euclid's elements.

George: That's great! And all you needed were parallel lines. But let me ask you, what makes two lines parallel?

Wellington: That's easy, two lines that don't intersect.

George: Here's how Euclid states it in his 11th Axiom: If a straight line (*C*) falling on two straight lines (*A* and *B*) makes the interior angles (*a* and *b*) on the same side less than two right angles

FIGURE 3



(180°), the two straight lines, if produced infinitely, meet on that side on which the angles are less than the two right angles (**Figure 3**).

Wellington: That's a pretty rigorous proof.

George: Or, the inverse which Euclid carefully avoids stating: If *a* and *b* are equal to 180° then *A* and *B* are said to be parallel, never to intersect.

Wellington: Accepted.

George: Let's construct this paradox, so it's very clear. Pull out some paper and draw it. Replicating the image, try it first with the angles *a* and *b* being small enough so that your lines *A* and *B* intersect and form a triangle on the paper.

Wellington: Easy enough, looks like they intersect to me.

George: All right, now start over, and draw another with angle *a* and *b* being a little wider. Do they eventually intersect?

Wellington: Looks good.

George: And once more; this time make it very wide, but not bigger than 179°. Did they cross?

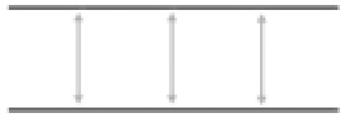
Wellington: No. Well, not yet.

George: Maybe you need another sheet of paper? . . . Try it with a huge piece of paper.

Wellington: Well, because it worked before, I can imagine it makes it there eventually.

George: Like this one here? (**Figure 4**)

FIGURE 4



Wellington: Yes, always maintaining this perpendicular relationship, the lines never get closer to each other; that's what makes them parallel.

George: Well, what about these two lines? They're everywhere the same distance from each other (**Figure 5**). With these, is our previous construction, shown in Figure 2, true? (**Figure 6**)

Wellington: Well, the lines have to be straight.

George: What does it mean for lines to be straight?

Wellington: It means that they're not curved.

George: What does it mean for a line to be curved? (**Figure 7**)

Wellington: If a line is straight, it will be the shortest distance between any two points. If it's at all curved, it will be longer than necessary to travel from one point to the other.

George: It's as if we were to walk from here directly to another city, without ever turning.

Wellington: Well, no. In that case the line would be curved, because you're not walking on a flat plane. The real shortest distance between any two points on the Earth would not be along the surface of the Earth, but along the flat plane cutting through the Earth.

George: And how would we know our flat plane was flat, when the Earth wasn't?

Wellington: The plane wouldn't be curved like the Earth. The plane would

FIGURE 5

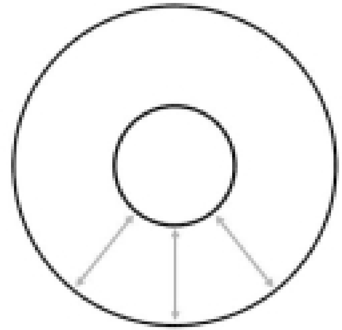
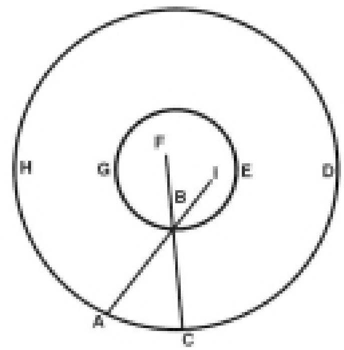


FIGURE 6



only be two-dimensional, while the Earth would be three-dimensional. You could walk everywhere on the plane by going forward and backward or left and right, without having to go up or down.

George: You mean to tell me that that's not true on the surface of the Earth? Do you need any other directions besides the two—North-South and East-West—when giving someone directions, for instance, or in navigating? How does the Earth not have two dimensions? Or any surface you're

Box 6 continues on next page

FIGURE 7

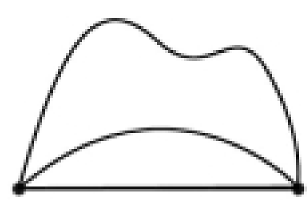
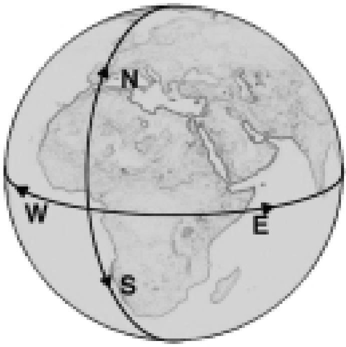


FIGURE 8

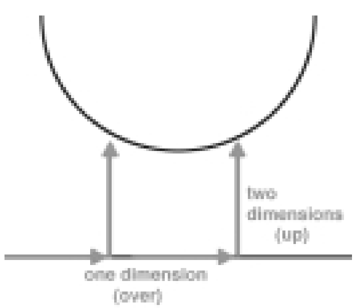


standing on for that matter? (**Figure 8**)

Wellington: No, curved surfaces involve a vertical motion as part of the other two motions. We'll use an example with lines instead of surfaces, which makes the same point. For the straight line, you only need to go one direction, over. But for the curved line you need to go over, and then up. You can get everywhere on the straight line with one dimension, but the curved line takes two. (**Figure 9**)

George: But you just drew "up" *relative* to a straight line. And we still don't know what a straight line or a flat plane is yet. What's more, if you took that picture and turned it upside down, we could say that the thing you called curved only went in one direction, North-South say, but that the distance from it of the thing you called flat was changing constantly. Over, and then up. By your definition, that would make the curved line one-dimensional, and

FIGURE 9



the flat line two-dimensional. (**Figure 10**)

Wellington: Wait, now I'm confused, this is even more bizarre than that story you started out with.

George: Well, it's exactly what Abraham Kästner said about the problem we're having: "Thus the difficulty concerns the distinction between curved and straight lines. A curved line means a line in which no part is straight. This concept of a curved line is *distinct,* because the concept of straight line is *clear;* but it is also *incomplete,* because the concept of straight line is *merely* clear."

It seems very clear to us what curved and straight are, and as a result we don't bother to ask the question. What we run into when we ask this question, is the debilitating brainwashing which was imposed on ancient Greek geometry by Euclid in creating his formal (prison) system. Kästner challenged this arbitrary authority, provoking his student, Carl Friedrich Gauss, to finally answer the question—"What is curvature?"—decisively.[2]

—*Sky Shields and Aaron Halevy*

1. "On the Conceptions that Underlie Space," by Abraham Kästner, 1790. A translation can be found of relevant paragraphs in *Fidelio* magazine, Spring/Summer 2004.

2. See the following source material:

"General Investigations of Curved Surfaces" by C.F. Gauss, 1827.

"Copenhagen Prize Essay" by C.F. Gauss, 1824.

Elements by Euclid, Dover Edition.

FIGURE 10