goals are achieved only on the condition that the student works through Gauss's own cognitive experience, both in making the discovery and in refuting reductionism generically. It is the inner, cognitive sense of "I know," rather than "I have been taught to believe," which must become the clearly understood principle of a revived policy of a universalized Classical humanist education.

Once a dedicated student achieves the inner cognitive sense of "I know this," he, or she has gained a bench-mark against which to measure many other things.

Bringing the Invisible To the Surface

by Bruce Director

This is the second half of a pedagogical exercise on the great mathematician Carl Gauss' delving into the Fundamental Theorem of Algebra—something all high school graduates think *they have learned. The first part, "The Fundamental Theorem: Gauss' Declaration of Independence," was published in* EIR *of April 12.*

When Carl Friedrich Gauss in 1798 criticized the state of mathematics for its "shallowness," he spoke literally; and not only about his time, but also ours. Then, as now, it had become popular for academics to ignore, and even ridicule, any effort to search for universal physical principles, restricting the province of scientific inquiry to the seemingly more practical task, of describing only what's visible on the surface. Ironically, as Gauss demonstrated in his 1799 doctoral dissertation on the fundamental theorem of algebra, what's on the surface is revealed only if one knows what's underneath.

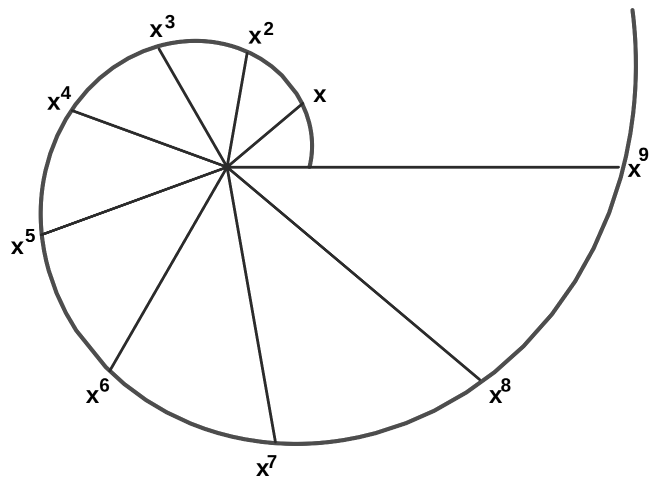
Gauss' method was ancient, made famous in Plato's metaphor of the cave, given new potency by Johannes Kepler's application of Nicholas of Cusa's method of *On Learned Ignorance.* For them, the task of the scientist was to bring into view, the underlying physical principles that could not be viewed directly—the unseen that guided the seen.

Take the case of Fermat's discovery of the principle, that refracted light follows the path of least time, instead of the path of least distance followed by reflected light. The principle of least distance is one that lies on the surface, and can be demonstrated in the visible domain. On the other hand, the principle of least time exists "behind," so to speak, the visible; brought into view only in the mind. On further reflection, it is clear, that the principle of least time was there all along, controlling, invisibly, the principle of least distance. In Plato's terms of reference, the principle of least time is of a "higher power" than the principle of least distance.

Fermat's discovery is a useful reference point for grasping Gauss' concept of the complex domain. As Gauss himself stated, unequivocally, the complex domain does not mean Euler's formal, superficial concept of "impossible" or imaginary numbers, as taught by "experts" since. Rather, Gauss' concept of the complex domain, like Fermat's principle of least time, brings to the surface, a principle that was there all along, but hidden from view.

As Gauss emphasized in his jubilee re-working of his 1799 dissertation, the concept of the complex domain is a "higher domain," independent of all *a priori* concepts of space. Yet, it is a domain, "in which one cannot move without the use of language borrowed from spatial images."

FIGURE 1



A succession of algebraic powers is generated by a self-similar spiral. For equal angles of rotation, the lengths of the corresponding radii are increased to the next power.

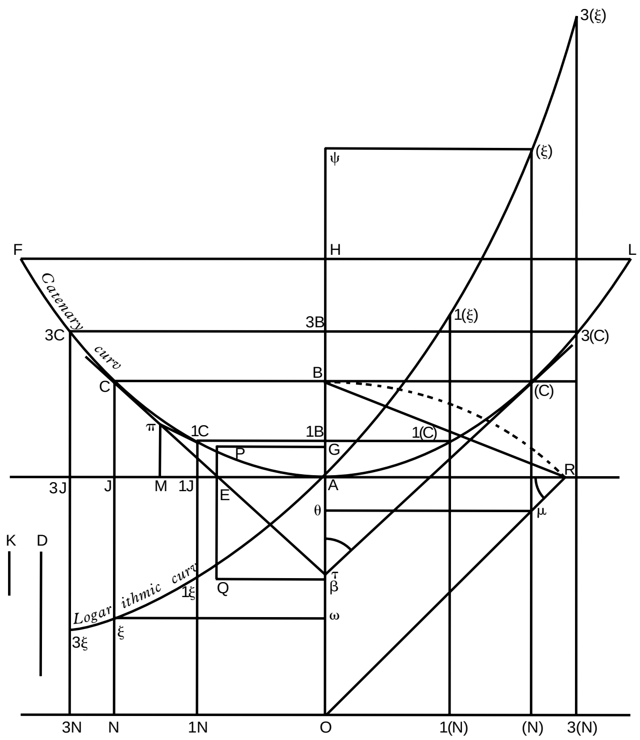
The Algebraic and the Transcendental

The issue for him, as for Gottfried Leibniz, was to find a general principle that characterized what had become known as "algebraic" magnitudes. These magnitudes, associated initially with the extension of lines, squares, and cubes, all fell under Plato's concept of *dunamais,* or *powers.*

Leibniz had shown, that while the domain of all "algebraic" magnitudes consisted of a succession of higher powers, this entire algebraic domain was itself dominated by a domain of a still higher power, which Leibniz called "transcendental." The relationship of the lower domain of algebraic magnitudes, to the higher non-algebraic domain of transcendental magnitudes, is reflected in what Jakob Bernoulli discovered about the equi-angular spiral (see **Figure 1**).

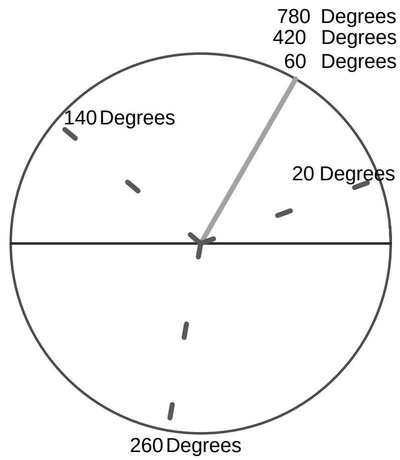
Leibniz, with Jakob's brother Johann Bernoulli, subsequently demonstrated that this higher, transcendental domain does not exist as a purely geometric principle, but originates from the physical action of a hanging chain, whose geometric

FIGURE 2



Leibniz' construction of the algebraic powers from the hanging chain, or catenary curve.

FIGURE 3



An example of the three solutions to the trisection of an angle.

shape Christiaan Huygens called a *catenary* (see **Figure 2**). Thus, the physical universe itself demonstrates that the "algebraic" magnitudes associated with extension, are not *generated* by extension. Rather, the algebraic magnitudes are generated from a physical principle that exists beyond simple extension, in the higher, transcendental, domain.

Gauss, in his proofs of the fundamental theorem of algebra, showed that even though this transcendental physical principle was outside the domain of the visible, it nevertheless cast a shadow that could be made visible in what Gauss called the complex domain.

As indicated in part one of this article, the discovery of a general principle for algebraic magnitudes was found, by looking through the "hole" represented by the square roots of negative numbers. These square roots appeared as solutions to algebraic equations, but lacked any apparent physical meaning. For example, in the algebraic equation $x^2=4$, x signifies the side of a square whose area is 4; while, in the equation $x^2=-4$, the x signifies the side of a square whose area is –4, an apparent impossibility.

For the first case, it is simple to see, that a line whose length is 2 would be the side of the square whose area is 4. However, from the standpoint of the algebraic equation, a line whose length is –2, also produces the desired square. At first glance, a line whose length is –2 seems as impossible as a square whose area is –4. Yet, if you draw a square of area 2, you will see that there are two diagonals, both of which have the power to produce a new square whose area is 4. These two magnitudes are distinguished from one another only by their direction, so one is denoted as 2 and the other as –2.

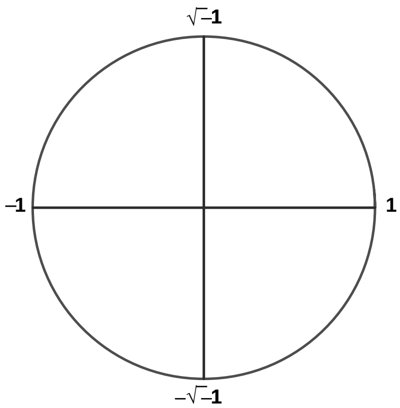
Now, extend this investigation to the cube. In the algebraic equation $x^3=8$, there appears to be only one number, 2, which satisfies the equation, and this number signifies the length of the edge of a cube whose volume is 8. This appears to be the only solution to this equation since $(-2)(-2)(-2)=-8$. The anomaly that there are two solutions, which appeared for the case of a quadratic equation, seems to disappear, in the case of the cube, for which there appears to be only one solution.

Trisecting an Angle

Not so fast. Look at another geometrical problem which, when stated in algebraic terms, poses the same paradox: the trisection of an arbitrary angle. Like the doubling of the cube, Greek geometers could not find a means for equally trisecting an arbitrary angle, from the principle of circular action itself. The several methods discovered (by Archimedes, Eratosthenes, and others), to find a general principle of trisecting an angle, were similar to those found, by Plato's collaborators, for doubling the cube. That is, this magnitude could not be constructed using only a circle and a straight line, but it required the use of extended circular action, such as conical action. But, trisecting an arbitrary angle presents another type of paradox which is not so evident in the problem of doubling the cube. To illustrate this, make the following experiment:

Draw a circle (**Figure 3**). For ease of illustration, mark off an angle of 60°. It is clear that an angle of 20° will trisect this angle equally. Now add one circular rotation to the 60° angle, making an angle of 420°. It appears these two angles

FIGURE 4



The unit of action in Gauss' complex domain.

are essentially the same. But, when 420° is divided by 3, we get an angle of 140°. Add another 360° rotation and we get to the angle of 780°, which appears to be exactly the same as the angles of 60° and 420°. Yet, when we divide 780° by 3 we get 260°. Keep this up, and you will see that the same pattern is repeated over and over again.

Looked at as a "sense certainty," the 60° angle can be trisected by only one angle, the 20° angle. Yet, when looked at beyond sense certainty, there are clearly three angles that "solve" the problem.

This illustrates another "hole" in the algebraic determination of magnitude. In the case of quadratic equations, there seem to be two solutions to each problem. In some cases, such as $x^2=4$, those solutions seem to have a visible existence; while for the case, $x^2=-4$, there are two solutions, $2\sqrt{-1}$ and $-2\sqrt{-1}$, both of which seem to be "imaginary," having no physical meaning. In the case of cubic equations, sometimes there are three visible solutions, such as in the case of trisecting an angle. But in the case of doubling the cube, there appears to be only one visible solution, but two "imaginary" solutions: $-1-(\sqrt{3})(\sqrt{-1})$; and $-1+(\sqrt{3})(\sqrt{-1})$.

Biquadratic equations, such as $x^4=16$, that seem to have no visible meaning themselves, have four solutions, two "real" (2 and –2) and two "imaginary" ($2\sqrt{-1}$ and $-2\sqrt{-1}$).

Things get even more confused for algebraic magnitudes of still higher powers. This anomaly poses the question that Gauss resolved in his proof of what he called the fundamental theorem of algebra: How many solutions are there for any algebraic equation?

The "shallow"-minded mathematicians of Gauss' day, such as Euler, Lagrange, and D'Alembert, took the superficial approach of asserting that any algebraic equation has as many solutions as it has powers, even if those solutions were "impossible," such as the square roots of negative numbers. (This sophist's argument is analogous to saying, "There is a difference between man and beast; but, this difference is meaningless.")

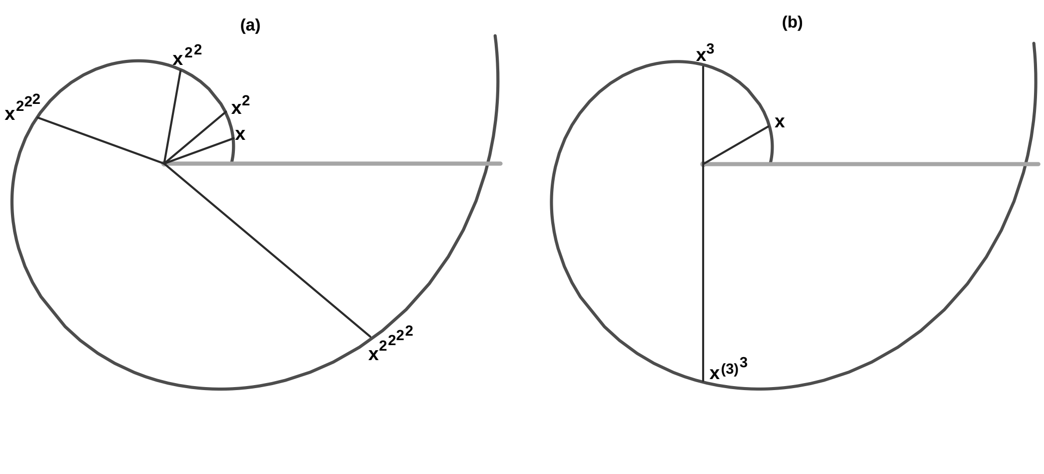
Shadows of Shadows: The Complex Domain

Gauss polemically exposed this fraud for the sophistry it was. "If someone would say a rectilinear equilateral right triangle is impossible, there will be nobody to deny that. But, if he intended to consider such an impossible triangle as a new species of triangles and to apply to it other qualities of triangles, would anyone refrain from laughing? That would be playing with words, or rather, misusing them."

For, Gauss, no magnitude could be admitted, unless its principle of generation was demonstrated. For magnitudes associated with the square roots of negative numbers, that principle was the complex physical action of *rotation, combined with extension.* Gauss called the magnitudes generated by this complex action, "complex numbers." Each complex number denoted a quantity of combined rotational, and extended action.

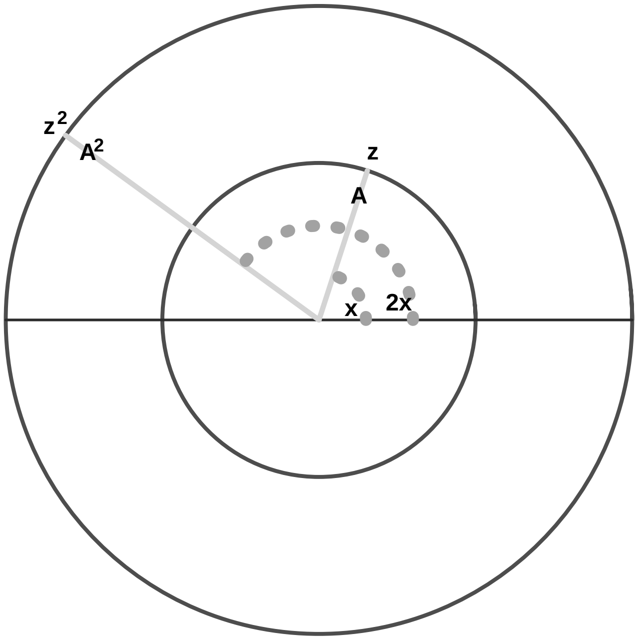
The unit of action in Gauss' complex domain is a circle, which is one rotation, with an extension of one (unit length).

FIGURE 5



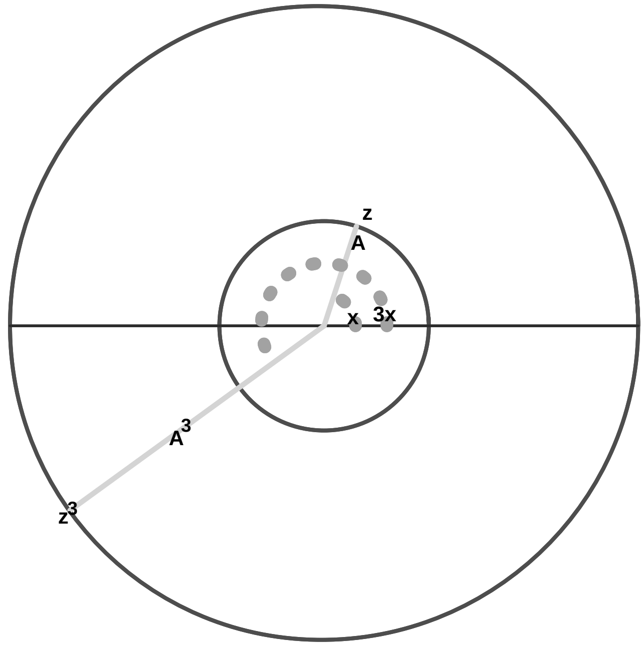
In (a) the lengths of the radii are squared as the angle of rotation doubles. In (b) the lengths of the radii are cubed as the angle of rotation triples.

FIGURE 6



Squaring a complex number.

FIGURE 7



Cubing a complex number.

In this domain, the number 1 signifies one complete rotation; -1, half a rotation; $\sqrt{-1}$, one-fourth of a rotation; and $-\sqrt{-1}$, three-fourths of a rotation (**Figure 4**).

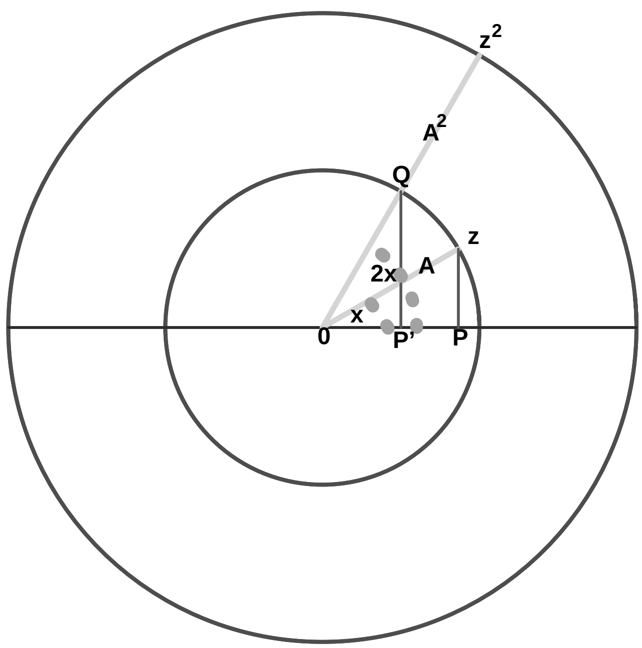
These "shadows of shadows," as he called them, were only a visible reflection of a still higher type of action, that was independent of all visible concepts of space. These higher forms of action, although invisible, could nevertheless be brought into view as a projection onto a surface.

Gauss' approach is consistent with that employed by the circles of Plato's Academy, as indicated by their use of the term *epiphanea* to indicate a surface (it comes from the same root as the word, "epiphany"). The concept indicated by the word *epiphanea* is, "that on which something is brought into view."

From this standpoint, Gauss demonstrated, in his 1799 dissertation, that the fundamental principle of generation of any algebraic equation, of no matter what power, could be brought into view, "epiphanied," so to speak, as a surface in the complex domain. These surfaces were visible representations, not—as in the cases of lines, squares, and cubes—of what the powers produced, but of the *principle* that produced the powers.

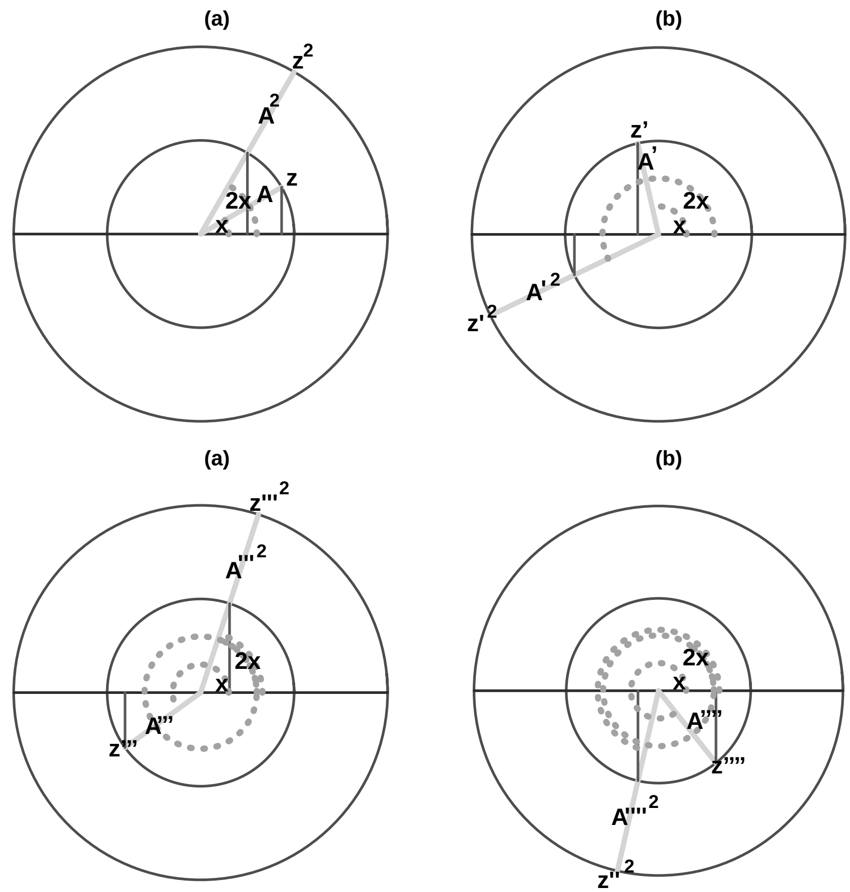
To construct these surfaces, Gauss went outside the simple visible representation of powers—such as squares and cubes—by seeking a more general form of powers, as exhibited in the equi-angular spiral (**Figure 5**). Here, the generation of a power, corresponds to the extension produced by an angular change. The generation of square powers, for example,

FIGURE 8



The sine of angle x is the line zP and the cosine of x is 0P. The sine of 2x is the line QP′ and the cosine is OP′.

FIGURE 9



Variations of the sine and cosine from the squaring of a complex number.

corresponds to the extension that results from a doubling of the angle of rotation, within the spiral (**5a**); and the generation of cubed powers corresponds to the extension that results from tripling the angle of rotation, within that spiral (**5b**). Thus, it is the *principle of squaring* that produces square magnitudes, and the *principle of cubing* that produces cubics.

In **Figure 6,** the complex number z is "squared" when the angle of rotation is doubled from x to $2x$ and the length squared from A to A^2. In doing this, the smaller circle maps twice onto the larger, "squared" circle. In **Figure 7,** the same principle is illustrated with respect to cubing. Here the angle x is tripled to $3x$, and the length A is cubed to A^3. In this case, the smaller circle maps three times onto the larger, "cubed" circle. And so on for the higher powers. The fourth power maps the smaller circle four times onto the larger. The fifth power, five times, and so forth.

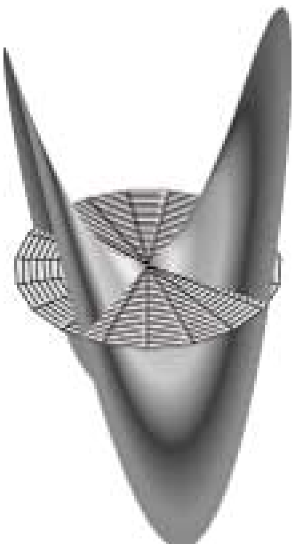
This gives a general principle that determines all algebraic powers: From this standpoint, all powers are reflected by the same action. The only thing that changes with each power, is the number of times that action occurs. Thus, each power is distinguished from the others, not by a particular magnitude, but by a topological characteristic.

In his doctoral dissertation, Gauss used this principle to generate surfaces that expressed the essential characteristic of powers in an even more fundamental way. Each rotation and extension produced a characteristic right triangle. The vertical leg of that triangle is the *sine* and the horizontal leg of that triangle is the *cosine* (**Figure 8**). There is a cyclical relationship between the sine and cosine which is a function of the angle of rotation. When the angle is 0, the sine is 0 and the cosine is 1. When the angle is 90°, the sine is 1 and the cosine is 0. Looking at this relationship for an entire rotation, the sine goes from 0 to 1 to 0 to –1 to 0; while the cosine goes from 1 to 0 to –1 to 0 and back to 1 (**Figure 9**).

In Figure 9, as z moves from 0 to 90°, the sine of the angle varies from 0 to 1; but at the same time, the angle for z^2 goes from 0 to 180°, and the sine of z^2 varies from 0 to 1 and back to 0. Then, as z moves from 90° to 180°, the sine varies from 1 back to 0, but the angle for z^2 has moved from 180° to 360°, and its sine has varied from 0 to –1 to 0. Thus, in one half rotation for z, the sine of z^2 has varied from 0 to 1 to 0 to –1 to 0. In his doctoral dissertation, Gauss represented this complex of actions as a surface (**Figures 10, 11, 12**). Each point on the surface is determined so that its height above the flat plane, is equal to the distance from the center, times the sine of the angle of rotation, as that angle is increased by the effect of the power. The *power* of any point in the flat plane, is represented by the height of the surface above that point. Thus, as the numbers on the flat surface move outward from the center, the surface grows higher according to the power. At the same time, as the numbers rotate around the center, the sine will pass from positive to negative. Since the numbers on the surface are the powers of the numbers on the flat plane, the number of times the sine will change from positive to negative, depends on how much the power multiplies the angle (double for square powers, triple for cubics, etc.). Therefore, each surface will have as many "humps" as the equation has dimensions. Consequently, a quadratic equation will have two "humps" up and two "humps" down (Figure 10). A cubic equation will have three "humps" up and three "humps" down. (Figure 11). A fourth-degree equation will have four "humps" in each direction (Figure 12); and so on.

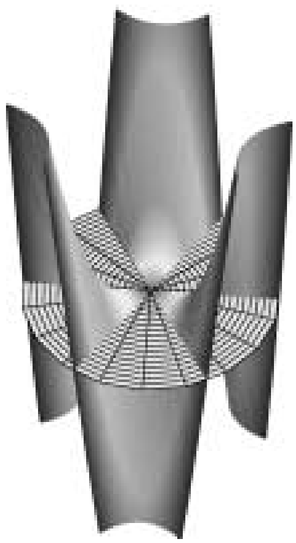
Gauss specified the construction of two surfaces for each algebraic equation, one based on the variations of the sine and the other based on the variations of the cosine (**Figure 13**). Each of these surfaces will define definite curves where the surfaces intersect the flat plane. The number of curves will depend on the number of "humps," which in turn depend on the highest power. Since each of these surfaces will be rotated 90° to each other, these curves will intersect each other, and the number of intersections will correspond to the number of powers (**Figure 14**). If the flat plane is considered to be zero, these intersections will correspond to the solutions, or "roots" of the equation. This proves that an algebraic equation has as many roots as its highest power.

FIGURE 10



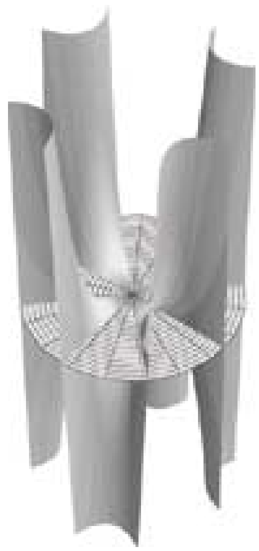
A Gaussian surface for the second power.

FIGURE 11



A Gaussian surface for the third power.

FIGURE 12



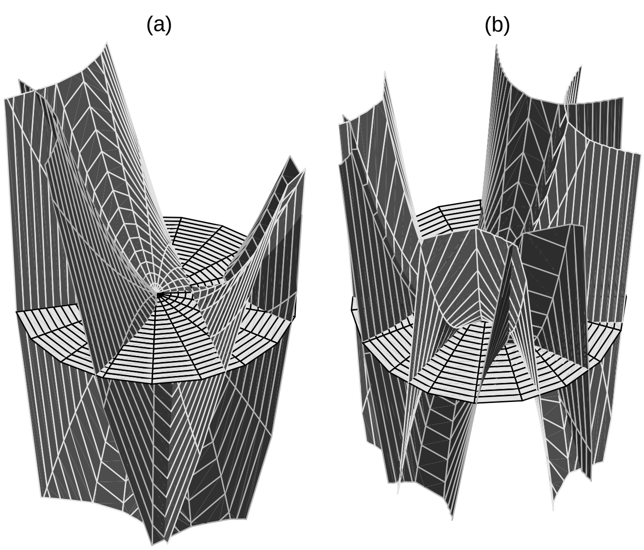
A Gaussian surface for the fourth power.

Step back and look at this work. These surfaces were produced, not from visible squares or cubes, but from the general principle of squaring, cubing, and higher powers. They represent, metaphorically, a principle that manifests itself physically, but cannot be seen. By projecting this principle—the general form of Plato's powers—onto these complex surfaces, Gauss has brought the invisible into view, and made intelligible what is incomprehensible in the superficial world of algebraic formalism.

The effort to make intelligible the implications of the complex domain was a focus for Gauss throughout his life. Writing to his friend Hansen on Dec. 11, 1825, Gauss said: "These investigations lead deeply into many others, I would even say, into the Metaphysics of the theory of space, and it is only with great difficulty can I tear myself away from the results that spring from it, as, for example, the true metaphysics of negative and complex numbers. The true sense of the square root of −1 stands before my mind fully alive, but it becomes very difficult to put it in words; I am always only able to give a vague image that floats in the air."

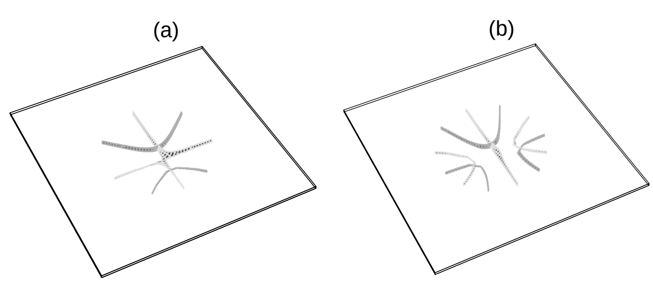
It was here, that Bernhard Riemann began.

FIGURE 13



(a) combines the surfaces based on the variations of the sine and cosine for the second power. (b) combines the surfaces based on the variations of the sine and cosine for the third power.

FIGURE 14



(a) is the intersection of the surfaces in 13(a) with the flat plane.
(b) is the intersection of the surfaces in 13(b) with the flat plane.