Box 1

Three Species of Number

Let's play a game! One player will geometrically construct two lengths by whatever means he chooses. Can the other player always determine how the lengths were created? In fact, can he ever? Maybe this is not a game worth playing!

A first hypothesis would be that the constructor took a certain length, and simply made two lines by replicating his length a whole number of times: for example, using — as our basic unit, we could create lengths by adding this line to itself, perhaps creating

— — — —

and

— — — — —

with the unit. These two lines have what the Pythagoreans called a rational relationship between themselves, expressed as the ratio 4-to-5, 4 : 5, or the familiar fraction 4/5. But how can we find the unit if the lines are not marked off already? An algorithm that will find the common line that made the two (*if* one exists!), operates by measuring the larger with the smaller and then using the remainder to attempt to measure the smaller original length:

For example, if we were the second player and were given the lengths:

— — — — — — — — — — —

and

— — — — — — —

We could measure the larger by the smaller:

— — — — — — —|— — — —

Which leaves a small remainder left over:

— — — —

Which can be used to measure the smaller original line:

— — — —|— — —

Now the line on the right has a remainder as well:

— — —

Now, measure again, this time measuring the left remainder with the right:

— — —|—

We now have a remainder on the left that can measure the remainder on the right:

—|—|—

Aha! Now all lines are accounted for and expressible, since they can be built up starting from this smallest unit magnitude. Try it with a friend!

Now, will it always happen that this technique succeeds? What if two magnitudes had no common, literal measure, and we could never find the common unit?

Take the case of the side of a square (*PQ*) and its diagonal (*PR*) (**Figure 1**). As Plato's *Meno* dialogue indicates, the diagonal is the solution for the creation of the doubled square, as the solution to a problem regarding *area*, not length. Here, the diagonal was not created by the simple addition of lines. The same technique of exhaustion applied above takes a new geometrical form with this example, which you should work through with a square cut out of paper.

Fold down the top line *PQ* onto the diagonal *PR* (**Figure 2**). *Q* will reach *T* and you will have a fold on your paper of *PV*. Looking at *PTR*, this is similar to the method with the lines above. We have cut line *PT* (of length *PQ*) out of hypotenuse *PR*, leaving behind remainder *TR*. But now something remarkable has happened. Since *TV* (and *TR*) are the same as *QV* in the construction, and the sides of a square are equal, $QR-QV$ is the same as $PQ-TR$, where *TR* is the remainder $PR-PQ$. This is analogous to measuring 7 with 4 above. But, look! The small remainder triangle *VTR* has exactly the same relationships as the original triangle *PQR*, so this process will never end! What does this imply? How small is our final, smallest unit, if it indeed exists?

Let's try again! What if we *had* found a common unit, what kind of ratio would the two lengths have? Well, if each length is made of a number of the unit, then it either could or could not be evenly divided in half producing whole units (it is either odd or even). Now if *PR* were odd, then the square that it makes would be made of an odd number of little unit

FIGURE 1

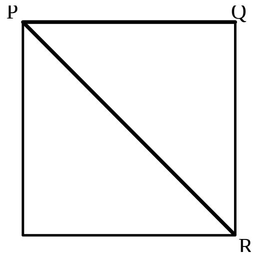


FIGURE 2

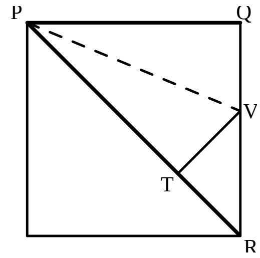
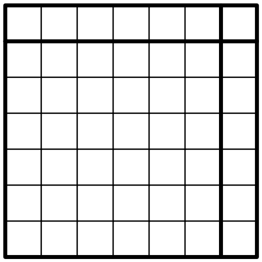


FIGURE 3

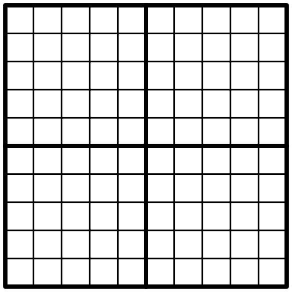


A square that is odd on each of its sides can be thought of as an even square with an L-shaped gnomon added to it. That gnomon is two even lines, with one square left over. That leftover square means that the entire odd-side square has an odd number of unit areas.

squares, but *PR* was supposed to make a square twice as big as *PQ,* and an odd number certainly isn't twice as big as anything, for odd means that it cannot be evenly divided in two (**Figure 3**)!

So, *PR* must be even in order to be twice the *PQ* square. Now if *PQ* were also even, it would mean that we got carried away in making our small unit, for a ratio of two even numbers is also a ratio with an odd number. For example, 2-to-3 could be 4-to-6 if you really wanted to call it that, just like one half is the same as two quarters. The only conclusion left is that *PR* is even, while *PQ* is odd, which makes the *PQ* square also have an odd number of small unit area squares. But wait, *PR* is even, which makes the *PR* square divisible this way (**Figure 4**):

FIGURE 4



Half the area of *PR* is even, but the *PQ* square, which is supposed to be half the *PR* square, is odd! We have failed again, and that was the last possibility. What does this mean? Is there really no possibility of a common unit? Then how can we express the relationship between these lengths?

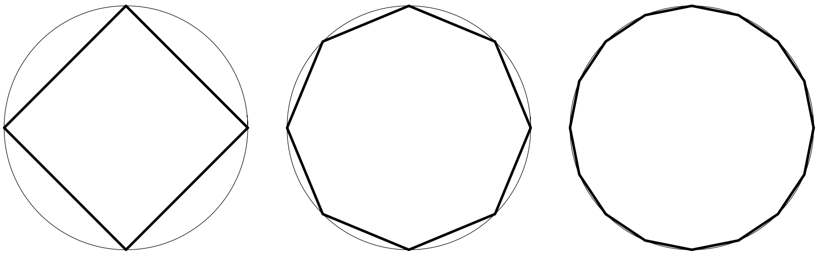
This is an *irrational* relationship: The side *PQ* and the diagonal *PR* of a square cannot both be expressed as a ratio countable by a common unit. But the inability to *express* a magnitude does not mean either that it is unknowable or unconstructable.

Theaetetus recounts, in Plato's *Theaetetus* dialogue, his concept of an entire class of such magnitudes: those that correspond to the sides of squares of commensurable areas, and to the sides of cubes of commensurable volumes. It should come to no surprise that the power to double a square or a cube, being of a higher *power* than that of doubling the line, is inexpressible in terms of lines.

The Transcendental Species

Beyond these two species, the rational and the irrational, exists the *transcendental.* Nicholas of Cusa's discussion of the quadrature of the circle (the exact measurement of the circumference of a circle in terms of its diameter) demonstrates this impossibility (**Figure 5**).

FIGURE 5



The attempt to approximate a circle by polygons of ever-increasing sides fails. Even at an astronomical number of sides on the polygon, each tiny side remains straight while the circle is curved in that interval. The failure of this approach demonstrates *negatively* that the circle is of a higher, *transcendental* species-type than the lines of the polygons with which we are attempting to reach it. It can be grasped only with a higher power, which Cusa named the isoperimetric ("Minimum-Maximum") principle.

The Kepler problem, arising as a distinction between irrationals and transcendentals, was a commission to future thinkers to develop a physical mathematics based on *power* as primary, rather than the non-physical hoax, which is only capable of expressing the effects of a power by the imagery of the tracks it leaves in its wake.

Riemann's surface functions, as elaborated in such locations as his *Theory of Abelian Functions,* more fully reveals the geometric implication of the existence of circular functions, which are infinitely powerful from the standpoint of the algebraic irrationals, and of forms of transcendentals of powers greater yet than the circular.

—*Jason Ross*